Probability Exercises

CHAPTER 1

Combinatorics

Problems

(1) Suppose a License plate must consist of 7 numbers of letter. How many license plates are there if
   (a) there can only be letters?
      - **Solution:** 26^7
   (b) the first three places are numbers and the last four are letters?
      - **Solution:**$10^3 \cdot 26^4$
   (c) the first three places are numbers and the last four are letters, but there can not be any repetitions in the same license plate?
      - **Solution:**$10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 \cdot 23$

(2) A school of 50 students has awards for the top math, english, history and science student in the school
   (a) How many ways can these awards be given if each student can only win one award?
      - **Solution:**$50 \cdot 49 \cdot 48 \cdot 47$
   (b) How many ways can these awards be given if students can win multiple awards?
      - **Solution:**50^4

(3) An iPhone password can be made up of any 4 digit combination.
   (a) How many different passwords are possible?
      - **Solution:**10^4
   (b) How many are possible if all the digits are odd?
      - **Solution:**5^4
   (c) How many can be made in which all digits are different or all digits are the same?
      - **Solution:**$10 \cdot 9 \cdot 8 \cdot 7 + 10$

(4) There is a class of 25 people made up of 11 guys and 14 girls.
   (a) How many ways are there to make a committee of 5 people?
      - **Solution:**$\binom{25}{5}$
   (b) How many ways are there to pick a committee of 5 of all girls?
      - **Solution:**$\binom{14}{5}$
   (c) How many ways are there to pick a committee of 3 girls and 2 guys?
      - **Solution:**$\binom{14}{3} \cdot \binom{11}{2}$

(5) If a student council contains 10 people, how many ways are there to elect a president, a vice president, and a 3 person prom committee from the group of 10 students?
   - **Solution:**$10 \cdot 9 \cdot \binom{8}{3}$

(6) Suppose you are organizing your textbooks on a book shelf. You have three chemistry books, 5 math books, 2 history books and 3 english books.
   (a) How many ways can you order the textbooks if you must have math books first, english books second, chemistry third, and history fourth?
      - **Solution:** $5!3!3!2!$
   (b) How many ways can you order the books if each subject must be ordered together?
      - **Solution:** $4!\,(5!3!3!2!)$

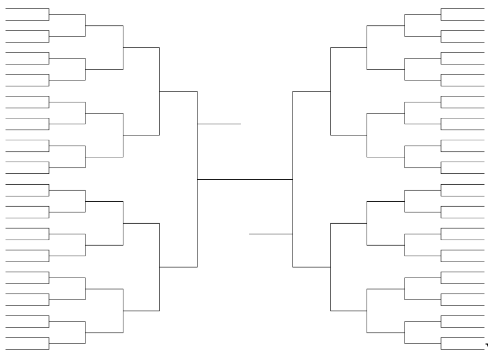
(7) You buy a Powerball lottery ticket. You choose 5 numbers between 1 and 59 (picked on white balls) and one number between 1 and 35 (picked on a red ball).How many ways can you
   (a) win the jackpot (guess all the numbers correctly)?
      - **Solution:** 1
   (b) match all the white balls but not the red ball?
      - **Solution:** $\binom{5}{5} \cdot 34$
   (c) match 3 white balls and the red ball?
      - **Solution:** $\binom{5}{3} \cdot \binom{54}{2} \cdot 1$

(8) A couple wants to invite their friends to be in their wedding party. The groom has 8 possible groomsmen and the bride has 11 possible bridesmaids. The wedding party will consist of 5 groomsman and 5 bridesmaids.
   (a) How many wedding party's are possible?
      - **Solution:** $\binom{8}{5} \cdot \binom{11}{5}$
   (b) Suppose that two of the possible groomsmen are feuding and will only accept an invitation if the other one is not going. How many wedding party's are possible?
      - **Solution:** $\binom{6}{5} \cdot \binom{11}{5} + \binom{2}{1} \cdot \binom{6}{4} \cdot \binom{11}{5}$
   (c) Suppose that two of the possible bridesmaids are feuding and will only accept an invitation if the other one is not going. How many wedding party's are possible?
      - **Solution:** $\binom{8}{5} \cdot \binom{9}{5} + \binom{8}{5} \cdot \binom{2}{1} \cdot \binom{9}{4}$
   (d) Suppose that one possible groosman and one possible woman refuse to serve together. How many wedding party's are possible?
      - **Solution:** $\binom{7}{5} \cdot \binom{10}{5} + 1 \cdot \binom{7}{4} \cdot \binom{10}{5} + \binom{7}{5} \cdot 1 \cdot \binom{10}{4}$

(9) There are 52 cards in a standard deck of playing cards. The poker hand is consists of five cards. How many poker hands are there?
   - **Solution:** $\binom{52}{5}$

(10) There are 30 people in a communications class. Each student must have a conversation with each student in the class for a project. How many total convesations will there be?
   - **Solution:** $\binom{30}{2}$

(11) Suppose a college basketball tournament consists of 64 teams playing head to head in a knock-out style tournament. There are 6 rounds, the round of 64, round of 32, round of 16, round of 8, the final four teams, and the finals. Suppose you are filling out a bracket such as this

which specifies which teams will win each game in each round. How many possible brackets can you make?

- **Solution:** First notice that the 64 teams play 63 total games: 32 games in the first round, 16 in the second round, 8 in the 3rd round, 4 in the regional finals, 2 in the final four, and then the national championship game.That is, 32+16+8+4+2+1= 63. Since there are 63 games to be played, and you have two choices at each stage in your bracket, there are 2^{63} different ways to fill out the bracket. That is

$$2^{63} = 9,223,372,036,854,775,808.$$

(12) You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?

- **Solution:** $\left(\begin{array}{c} 8 \\ 3,2,3 \end{array} \right) = \frac{8!}{3!2!3!}$.

CHAPTER 2

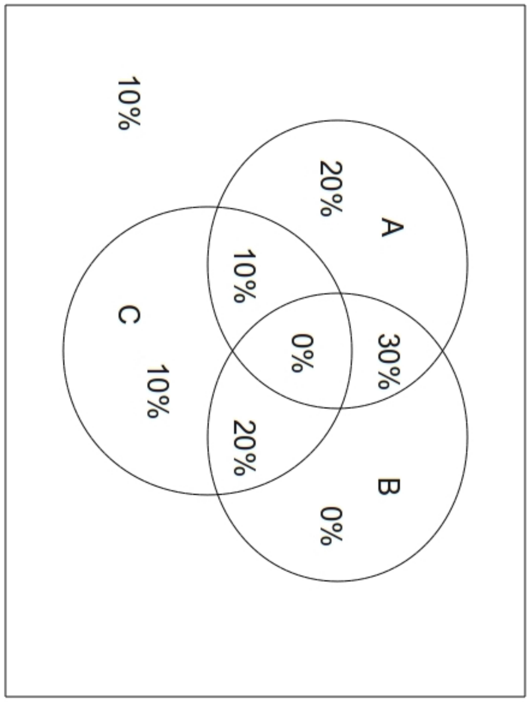
The probability set up

(1) Suppose a box contains 3 balls : 1 red, 1 green, and 1 blue
   (a) Consider an experiment that consists of taking 1 ball from the box and then replacing it in the box and drawing a second ball from the box. List all possible outcomes.
      - **Solution:** Since every marble can be drawn firrst and every marble can be drawn second, there are $32 = 9$ possibilities: RR, RG, RB, GR, GG, GB, BR, BG, and BB (we let the first letter of the color of the drawn marble represent the draw).
   (b) Consider an experiment that consists of taking 1 ball from the box and then drawing a second ball from the box without replacing the first. List all possible outcomes.
      - **Solution:** In this case, the color of the second marble cannot match the color of the rst, so there are 6 possibilities: RG, RB, GR, GB, BR, and BG.

(2) Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$.
   (a) What is the probability that A occurs but B does not?
      - **Solution:** Since A and B are mutually exclusive, the only way A can occur is when B does not. This means that$P(A \cap B^c) = P(A) = .3$.
   (b) What is the probability that neither A nor B occurs?
      - **Solution:** Since $A \cap B = \emptyset$. Axiom 3 tell us that $P(A \cup B) = P(A) + P(B) = .8$. Since we want $P(A^c \cap B^c)$, we use DeMorgan's law to see that this is $P(A^c \cap B^c) = P\left((A \cup B)^c\right) = 1 - \mathbb{P}(A \cup B) = .2$.

(3) Forty percent of college students from a certain college are members of neither an academic club nor a greek organization. Fifty percent are members of academic clubs while thirty percent are members of a greek organization. Suppose a student is chosen at random, what is the probability that this students is a member
   (a) of an academic club or a greek organization?
      - **Solution:** $\mathbb{P}(A \cup B) = 1 - \mathbb{P}\left((A \cup B)^c\right) = 1 - .4 = .6$
   (b) of an academic club and a greek organization?
      - **Solution:** $.6 = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = .5 + .3 - \mathbb{P}(A \cap B)$ Thus $\mathbb{P}(A \cap B) = .2$.

(4) In City, 60% of the households subscribe to newspaper A, 50% to newspaper B, 40% to newspaper C, 30% to A and B, 20% to B and C, and 10% to A and C, but none subscribe to all three. (Hint: Draw a Venn diagram)
   (a) What percentage subscribe to exactly one newspaper?
      - **Solution:** We use these percentages to produce the Venn diagram below:



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- This tells us that 30% of households subscribe to exactly one paper.

(b) What percentage subscribe to at most one newspaper?

- **Solution:** The Venn diagram tells us that 100% - (10% + 20% + 30%) = 40% of households subscribe to at most one paper.

(5) There are 52 cards in a standard deck of playing cards. The *poker hand* is consists of five cards. There are 4 *suits*: heats, spades, diamonds, and clubs (♡♠♢♣). The suit's diamonds and clubs are red while clubs and spades are black. In each suit there are 13 *ranks*: the numbers $2, 3 \ldots, 10$, the face cards, Jack, Queen, King, and the Ace. Find the probability of randomly drawing the following poker hands.

(a) All red cards?

- **Solution:** $\dfrac{\binom{26}{5}}{\binom{52}{5}}$

(b) Exactly two 10's and exactly three aces?

- **Solution:** $\dfrac{\binom{4}{2} \cdot \binom{4}{3}}{\binom{52}{5}}$

(c) all face cards or no face cards?

- **Solution:** $\dfrac{\binom{12}{5}}{\binom{52}{5}} + \dfrac{\binom{40}{5}}{\binom{52}{5}}$

(6) Find the probability of randomly drawing the following poker hands.
   (a) A *one pair,* which consists of two cards of the same rank and three other distinct ranks. (e.g. 22Q59)
   - **Solution:** $\binom{13}{1}\binom{12}{3}\binom{4}{2}\binom{4}{1}\binom{4}{1}\binom{4}{1} \Big/ \binom{52}{5}$

   (b) A *two pair*, which constists of two cards of the same rank, two cards of another rank, and another card of yet another rank. (e.g. JJ779)
   - **Solution:** $\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1} \Big/ \binom{52}{5}$

   (c) A *three of a kind*, which consists of a three cards of the same rank, and two others of distinct rank. (e.g. 4449K)
   - **Solution:** $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1} \Big/ \binom{52}{5}$

   (d) A *flush*, which consists of all five cards of the same suit. (e.g. HHHH, SSSS, DDDD, or CCCC)
   - **Solution:** $4\binom{13}{5} \Big/ \binom{52}{5}$

   (e) A *full house*, which consists of a two pair and a three of a kind. (e.g. 88844) (Hint: Note that 88844 is a different hand than a 44488)
   - **Solution:** $13 \cdot 12\binom{4}{3}\binom{4}{2} \Big/ \binom{52}{5}$

(7) Suppose a standard deck of cards is modifed with the additional rank of *Super King* and the additional suit of *Swords* so now each card has one of 14 ranks and one of 5 suits.
   (a) If a card is selected at random, what is the probability that it's the *Super King of Swords*.
   - **Solution:** $\frac{1}{70}$

   (b) What's the probability of getting a six card hand with exactly three pairs (two cards of one rank and two cards of another rank and two cards of yet another rank, e.g. 7,7,2,2,J,J) ?
   - **Solution:** $\binom{14}{3}\binom{5}{2}\binom{5}{2}\binom{5}{2} \Big/ \binom{70}{6}$

   (c) What's the probability of getting a six card hand which constists of three cards of the same rank, two cards of another rank, and another card of yet another rank. (e.g. 3,3,3,A,A,7)?
   - **Solution:** $14\binom{5}{3}13\binom{5}{2}12\binom{5}{1} \Big/ \binom{70}{6}$

   (d) What's the probability of getting a six card hand with exactly 2 pairs, and 2 singles (two cards of one rank and 1 card of another rank and 1 card of yet another rank, e.g. 7,7,2,2,J,K) ?
   - **Solution:** $\binom{14}{2}\binom{12}{2}\binom{5}{2}\binom{5}{2}\binom{5}{1}\binom{5}{1} \Big/ \binom{70}{6}$

(8) A pair of fair dice is rolled. What is the probability that the first die lands on a strictly higher value than the second die.

- **Solution:** Simple inspection we can see that the only possibilities

$$\begin{array}{rl} (6,1)\,(6,2)\,(6,3)\,(6,4)\,(6,5) & \text{5 possibilities} \\ (5,1)\,(5,2)\,(5,3)\,(5,4) & \text{4 possibilities} \\ (4,1)\,,(4,2),(4,3) & \text{3 possibilities} \\ (3,1)\,,(3,2) & \text{2 possibilities} \\ (2,1) & \text{1 possibility} \\ & = 15 \text{ total} \end{array}$$

Thus the probability is $\frac{15}{36}$.

(9) There are 8 students in a class. What is the probability that at least two students share a common birthday month?
- **Solution:** $1 - \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{12^8}$

(10) Nine balls are randomly withdrawn from an urn that contains 10 blue, 12 red, and 15 green balls. What is the probability that
(a) 2 blue, 5 red, and 2 green balls are withdrawn
- **Solution:** $\dfrac{\binom{10}{2}\binom{12}{5}\binom{15}{2}}{\binom{37}{9}}$

(b) at least 2 blue balls are withdrawn.
- **Solution:** $1 - \dfrac{\binom{27}{9}}{\binom{37}{9}} - \dfrac{\binom{10}{1}\binom{27}{8}}{\binom{37}{9}}$

(11) Suppose 4 valedictorians (from different high schools) were all accepted to the 8 Ivy League universities. What is the probability that they each choose to go to a different Ivy League university?
- **Solution:** $\frac{8 \cdot 7 \cdot 6 \cdot 5}{8^4}$

(12) Let S be a sample space with probability $\mathbb{P}$. Let E, F be any events in S. Using the Axioms of Probability or Proposition 1 from the Lecture notes (Section 2.2),
(a) Show that $\mathbb{P}(E \cap F^c) = \mathbb{P}(E) - \mathbb{P}(E \cap F)$:
- **Solution:** Helps to draw a Venn Diagram, and split the set into disjoint parts.
  - After drawing a Venn Diagram, you'll notice that we can write the set E into two parts:

$$E = (E \cap F^c) \bigcup (E \cap F)$$

where the sets $E \cap F^c$ and $E \cap F$ are disjoint.
  - Therefore by the disjoint property of probability (Proposition 1b, or Axiom 2) we have

$$\begin{aligned} \mathbb{P}(E) &= \mathbb{P}\left((E \cap F^c) \bigcup (E \cap F)\right) \\ &= \mathbb{P}(E \cap F^c) + \mathbb{P}(E \cap F) \end{aligned}$$

and moving things around you get $\mathbb{P}(E \cap F^c) = \mathbb{P}(E) - \mathbb{P}(E \cap F)$.

(b) Show *Bonferroni's inequality*: $\mathbb{P}(E \cap F) \geq \mathbb{P}(E) + \mathbb{P}(F) - 1$:

- **Solution:** Note that by Proposition 1e, we know that
$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F),$$
moving this inequality around we have
$$\mathbb{P}(E \cap F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cup F).$$
  - By Proposition 1d, since $E \cup F \subset S$ then $\mathbb{P}(E \cup F) \leq \mathbb{P}(S) = 1$. But since $\mathbb{P}(E \cup F) \leq 1$ then $-\mathbb{P}(E \cup F) \geq -1$. Plugging this into the equation above we have
$$\begin{aligned} \mathbb{P}(E \cap F) &= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cup F) \\ &\geq \mathbb{P}(E) + \mathbb{P}(F) - 1, \end{aligned}$$
which is exactly what we wanted to show.

CHAPTER 3

Independence

(1) Let A and B be two independent events with $P(A) = .4$ and $P(A \cup B) = .64$. What is $P(B)$?
- **Solution:** Using independence we have $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B)$ and substituting we have
$$.64 = .4 + \mathbb{P}(B) - .4\mathbb{P}(B).$$
Solving for $\mathbb{P}(B)$ we have $\mathbb{P}(B) = .4$.

(2) In a class, there are 4 male math majors, 6 female math majors, and 6 male actuarial science majors. How many actuarial science girls must be present in the class if sex and major are independent when choosing a student selected at random?
- **Solution:** Let x be the number of actuarial science girls. Then
$$\begin{aligned} \mathbb{P}(Boy \cap Math) &= \frac{4}{16+x}, \\ \mathbb{P}(Boy) &= \frac{10}{16+x} \\ \mathbb{P}(Math) &= \frac{10}{16+x}. \end{aligned}$$
Then using independence $\mathbb{P}(Boy \cap Math) = \mathbb{P}(Boy)\,\mathbb{P}(Math)$ so that
$$\frac{4}{16+x} = \frac{10^2}{(16+x)^2} \implies 4 = \frac{100}{16+x}$$
and solving for x we have $x = 9$.

(3) An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Calculate the number of blue balls in the second urn.
- **Solution:** Let R_i = even that a red ball is drawn from urn i and let B_i =event that a blue ball is drawn from urn i.
  - Let x be the number of blue balls in urn 2,
  - Then
$$\begin{aligned} .44 &= \mathbb{P}\Big((R_1 \cap R_2) \bigcup (B_1 \cap B_2)\Big) = \mathbb{P}(R_1 \cap R_2) + \mathbb{P}(B_1 \cap B_2) \\ &= \mathbb{P}(R_1)\,\mathbb{P}(R_2) + \mathbb{P}(B_1)\,\mathbb{P}(B_2) \\ &= \frac{4}{10}\frac{16}{x+16} + \frac{6}{10}\frac{x}{x+16}. \end{aligned}$$
  - Solve for x! You will get $x = 4$.

(4) Using only the definition of independence and any properties you already know about events/sets and probability to prove that if E and F are independent then E^c and F^c must also be independent.

- **Solution:** We want to show that $\mathbb{P}(E^c \cap F^c) = \mathbb{P}(E^c)\,\mathbb{P}(F^c)$ by only using properties of Probability that we know so far and the definition of independence. We are already given that $\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)$.
  - Now start by using a Venn Diagram to rewrite the event $E^c \cap F^c$ into disjoint parts. In fact, you'll see from the diagram that $E^c \cap F^c = (E \cup F)^c$ by deMorgan's law, which simply is everything outside of $E \cup F$. Hence

$$\begin{aligned}
\mathbb{P}(E^c \cap F^c) &= \mathbb{P}\left((E \cup F)^c\right) \\
&= 1 - \mathbb{P}(E \cup F) \\
&= 1 - \left[\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)\right] \\
&= 1 - \left[\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E)\,\mathbb{P}(F)\right], \text{ by independence of } E \text{ and } F \\
&= 1 - \mathbb{P}(E) - \mathbb{P}(F) + \mathbb{P}(E)\,\mathbb{P}(F) \\
&= \mathbb{P}(E^c) - \mathbb{P}(F) + \mathbb{P}(E)\,\mathbb{P}(F) \\
&= \mathbb{P}(E^c) - (1 - \mathbb{P}(E))\,\mathbb{P}(F) \\
&= \mathbb{P}(E^c) - \mathbb{P}(E^c)\,\mathbb{P}(F) \\
&= \mathbb{P}(E^c)\,(1 - \mathbb{P}(F)) \\
&= \mathbb{P}(E^c)\,\mathbb{P}(F^c),
\end{aligned}$$

  as needed to show.

CHAPTER 4

Conditional Probability

(1) Two dice are rolled. Let $A = \{\text{sum of two dice equals 3}\}$, $B = \{\text{sum of two dice equals 7 }\}$, and $C = \{\text{at least one of the dice shows a 1}\}$.

(a) What is $\mathbb{P}(A \mid C)$?

- **Solution:** Note that the sample space is $S = \{(i,j) \mid i,j = 1,2,3,4,5,6\}$ with each outcome equally likely. Then

$$\begin{aligned} A &= \{(1,2),(2,1)\} \\ B &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \\ C &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\} \end{aligned}$$

Then

$$\mathbb{P}(A \mid C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

(b) What is $\mathbb{P}(B \mid C)$?

- **Solution:**

$$\mathbb{P}(B \mid C) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

(c) Are A and C indepedent? What about B and C?

- **Solution:**Note that $\mathbb{P}(A) = 2/36 \neq \mathbb{P}(A \mid C)$, so they are not independent. Similarly, $\mathbb{P}(B) = 6/36 \neq \mathbb{P}(B \mid C)$, so they are not independent.

(2) Suppose you roll a two standard, fair, 6-sided dice. What is the probability that the sum is at least 9 given that you rolled at least one 6?

- **Solution:** Let E be the event “there is at least one 6" and F be the event }the sum is at least 9” We want to calculate $\mathbb{P}(F \mid E)$. Begin by noting that there are 36 possible rolls of these two dice and all of them are equally likely. We can see that 11 different rolls of these two dice will result in at least one 6, so $\mathbb{P}(E) = \frac{11}{36}$. There are 7 different rolls that will result in at least one 6 and a sum of at least 9 . They are $\{(6,3),(6,4),(6,5),(6,6),(3,6),(4,6),(5,6)\}$, so $\mathbb{P}(E \cap F) = \frac{7}{36}$. This tells us that

$$\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)} = \frac{7/36}{11/36} = \frac{7}{11}.$$

(3) Suppose that Annabelle and Bobby each draw 13 cards from a standard deck of 52. Given that Sarah has exactly two aces, what is the probability that Bobby has exactly one ace?

- **Solution:** Let A be the event “Annabelle has two aces," and let B be the event “Bobby has exactly one ace." Again, we want $\mathbb{P}(B \mid A)$, so we calculate $\mathbb{P}(A)$ and $\mathbb{P}(A \cap B)$. Annabelle could have any of $\binom{52}{13}$ possible hands. Of these hands, $\binom{4}{2} \cdot \binom{48}{11}$ will have exactly two aces, so

$$\mathbb{P}(A)=\frac{\binom{4}{2}\cdot\binom{48}{11}}{\binom{52}{13}}.$$

Now the number of ways in which Annabelle can have a certain hand and Bobby can have a certain hand is $\binom{52}{13}\cdot\binom{39}{13}$, and the number of ways in which A and B can both occur is $\binom{4}{2}\cdot\binom{48}{11}\cdot\binom{2}{1}\cdot\binom{37}{12}$. so

$$\mathbb{P}(A\cap B)=\frac{\binom{4}{2}\cdot\binom{48}{11}\cdot\binom{2}{1}\cdot\binom{37}{12}}{\binom{52}{13}\cdot\binom{39}{13}}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(B\mid A) &= \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}=\frac{\dfrac{\binom{4}{2}\cdot\binom{48}{11}\cdot\binom{2}{1}\cdot\binom{37}{12}}{\binom{52}{13}\cdot\binom{39}{13}}}{\dfrac{\binom{4}{2}\cdot\binom{48}{11}}{\binom{52}{13}}}\\ &= \frac{\binom{2}{1}\cdot\binom{37}{12}}{\binom{39}{13}}.\end{aligned}$$

(4) Color blindness is a sex-linked condition, and 5% of men and 0.25% of women are color blind. The population of the United States is 51% female. What is the probability that a color-blind American is a man?

- **Solution:** Let M be the event "an American is a man" and let$C$ be the event "" an American is color blind.". Then

$$\begin{aligned}\mathbb{P}(M\mid C) &= \frac{\mathbb{P}(C\mid M)\,\mathbb{P}(M)}{\mathbb{P}(C\mid M)\,\mathbb{P}(M)+\mathbb{P}(C\mid M^c)\,\mathbb{P}(M^c)}\\ &= \frac{(.05)\,(.49)}{(.05)\,(.49)+(.0025)\,(.51)}\approx .9505.\end{aligned}$$

(5) Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available.

(a) What is the chance that a purchased bulb will work for longer than 5000 hours?

- **Solution:** Let H be the event "works over 5000 hours". Let X be the event comes from factory X" and Y be the event "comes fom factory Y". Then by the Law of Total Probability

$$\begin{aligned} \mathbb{P}(H) &= \mathbb{P}(H \mid X)\,\mathbb{P}(X) + \mathbb{P}(H \mid Y)\,\mathbb{P}(Y) \\ &= (.99)(.6) + (.95)(.4) \\ &= .974. \end{aligned}$$

(b) Given that a lightbulb works for more than 5000 hours, what is the probability that it came from factory Y?
- **Solution:** By Part (a) we have

$$\begin{aligned} \mathbb{P}(Y \mid H) &= \frac{\mathbb{P}(H \mid Y)\,\mathbb{P}(Y)}{\mathbb{P}(H)} \\ &= \frac{(.95)(.4)}{.974} \approx .39. \end{aligned}$$

(c) Given that a lightbulb work does not work for more than 5000 hours, what is the probability that it came from factory X?
- **Solution:** We again use the result from Part (a)

$$\begin{aligned} \mathbb{P}(X \mid H^c) &= \frac{\mathbb{P}(H^c \mid X)\,\mathbb{P}(X)}{\mathbb{P}(H^c)} = \frac{\mathbb{P}(H^c \mid X)\,\mathbb{P}(X)}{1 - \mathbb{P}(H)} \\ &= \frac{(1 - .99)(.6)}{1 - .974} = \frac{(.01)(.6)}{.026} \\ &\approx .23 \end{aligned}$$

(6) A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from Machine A?
- **Solution:** Let $D = \{\text{Bolt is defective}\}$, $A = \{\text{bolt is form machine } A\}$, $B = \{\text{bolt is from machine } C\}$. Then by Baye's theorem

$$\begin{aligned} \mathbb{P}(A \mid D) &= \frac{\mathbb{P}(D \mid A)\,\mathbb{P}(A)}{\mathbb{P}(D \mid A)\,\mathbb{P}(A) + \mathbb{P}(D \mid B)\,\mathbb{P}(B) + \mathbb{P}(D \mid C)\,\mathbb{P}(C)} \\ &= \frac{(.05)(.25)}{(.05)(.25) + (.04)(.35) + (.02)(.4)} \\ &= .362. \end{aligned}$$

(7) A multiple choice exam has 4 choices for each question. A student has studied enough so that the probability they will know the answer to a question is 0.5, the probability that they will be able to eliminate one choice is 0.25, otherwise all 4 choices seem equally plausible. If they know the answer they will get the question right. If not they have to guess from the 3 or 4 choices. As the teacher you want the test to measure what the student knows. If the student answers a question correctly what's the probability they knew the answer?

- **Solution:** Let C be the vent the students the problem correct and K the event the students knows the answer. Using Bayes' theorem we have

$$
\begin{aligned}
&P(K \mid C) \\
&= \frac{P(C \mid K)P(K)}{P(C)} \\
&= \frac{P(C \mid K)P(K)}{P(C \mid K)P(K) + P(C \mid \text{Eliminates})P(\text{Eliminates}) + P(C \mid \text{Guess})P(\text{Guess})} \\
&= \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4}} = \frac{24}{31} \approx .774 = 77.4\%.
\end{aligned}
$$

(8) A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not actually present. One percent of the population actually has the disease. Calculate the probability that a person actually has the disease given that the test indicates the presence of the disease.

- **Solution:** Let + signiffy a positive test result, and D means dissease is present. Then

$$
\begin{aligned}
\mathbb{P}(D \mid +) &= \frac{\mathbb{P}(+ \mid D)\, P(D)}{P(+ \mid D)\, P(D) + P(+ \mid D^c)P(D^c)} \\
&= \frac{(.95)\,(.01)}{(.95)\,(.01) + (.005)\,(.99)} \\
&= .657.
\end{aligned}
$$

CHAPTER 5

Random Variables

(1) Two balls are chosen randomly from an urn containing 8 white balls, 4 black, and 2 orange balls. Supose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings.
  (a) What are the possible values of X?
   - **Solution:** Note that $X = -2, -1, -0, 1, 2, 4$.

  (b) What are the probabilities associated to each value?
   - **Solution:** Then

$$\mathbb{P}(X=4) = \mathbb{P}(BB) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91} \qquad \mathbb{P}(X=0) = \mathbb{P}(OO) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

$$\mathbb{P}(X=2) = \mathbb{P}(BO) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91} \qquad \mathbb{P}(X=-1) = \mathbb{P}(WO) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$

$$\mathbb{P}(X=1) = \mathbb{P}(BW) = \frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91} \qquad \mathbb{P}(X=-2) = \mathbb{P}(WW) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$

(2) A card is drawn at random from a standard deck of playing cards. If it is a heart, you win \$1. If it is a diamond, you have to pay \$2. If it is any other card, you win \$3. What is the expected value of your winnings?
 - **Solution:**

$$\mathbb{E}X = 1 \cdot \frac{1}{4} + (-2)\frac{1}{4} + 3 \cdot \frac{1}{2} = \frac{5}{4}$$

(3) The game of roulette consists of a small ball and a wheel with 38 numbered pockets around the edge that includes the numbers $1-36, 0$ and 00. As the wheel is spun, the ball bounces around randomly until it settles down in one of the pockets.
  (a) Suppose you bet \$1 on a single number and random variable X represents the (monetary) outcome (the money you win or lose). If the bet wins, the payoff is \$35 and you get your money back. If you lose the bet then you lose your \$1. What is the expected profit on a 1 dollar bet ?
   - **Solution:** The expexted profit is $\mathbb{E}X = 35 \cdot \left(\frac{1}{38}\right) - 1 \cdot \frac{37}{38} = -\$.0526$.

  (b) Suppose you bet \$1 on the numbers $1-18$ and random variable X represents the (monetary) outcome (the money you win or lose). If the bet wins, the payoff is \$1 and you get your money back. If you lose the bet then you lose your \$1. What is the expected profit on a 1 dollar bet ?

- **Solution:** If you will then your profit will be \$1. If you lose then you lose your \$1 bet. The expexted profit is $\mathbb{E}X = 1 \cdot \left(\frac{18}{38}\right) - 1 \cdot \frac{20}{38} = -\$.0526$.

(4) An insurance company finds that Mark has a 8% chance of getting into a car accident in the next year. If Mark has any kind of accident then the company guarantees to pay him \$10,000. The company has decided to charge Mark a \$200 premium for this one year insurance policy.

(a) Let X be the amount profit or loss from this insurance policy in the next year for the insurance company. Find $\mathbb{E}X$, the expected return for the Insurance company? Should the insurance company charge more or less on it's premium?

- **Solution:** If Mark has no accident then the company makes a profit of 200 dollars. If Mark has an accident they have to pay him 10,000 dollars, but regardless they received 200 dollars from him as an yearly premium. We have

$$\mathbb{E}X = (200 - 10,000) \cdot (.08) + 200 \cdot (.92) = -600.$$

On average the company will lose \$600 dollars. Thus the company should charge more.

(b) What amount should the insurance company charge Mark in order to guarantee an expected return of \$100?

- **Solution:** Let P be the premium. Then in order to guarantee an expected return of 100 then

$$100 = \mathbb{E}X = (P - 10,000) \cdot (.08) + P \cdot (.92)$$

and solving for P we get $P = \$900$.

(5) A random variable X has the following probability mass function: $p(0) = \frac{1}{3}$, $p(1) = \frac{1}{6}$, $p(2) = \frac{1}{4}$, $p(3) = \frac{1}{4}$. Find its expected value, variance, and standard deviation.

- **Solution:** Lets apply the formulas

$$\mathbb{E}X = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{34}{24}.$$

Now to calculate variance we have

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 \\
&= \left(0^2 \cdot \frac{1}{3} + 1^1\frac{1}{6} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4}\right) - \left(\frac{34}{24}\right)^2 \\
&= \frac{82}{24} - \frac{34^2}{24^2} \\
&= \frac{812}{24^2}.
\end{aligned}$$

Taking the square root gives us

$$\mathrm{SD}(X) = \frac{2\sqrt{203}}{24}.$$

(6) Suppose X is a random variable such that $\mathbb{E}[X] = 50$ and $\mathrm{Var}(X) = 12$. Calculate the following quantities.

(a) $\mathbb{E}\left[X^2\right]$

- **Solution:** Since $\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = 12$ then

$$\mathbb{E}\left[X^2\right] = \mathrm{Var}(X) + (\mathbb{E}X)^2 = 12 + 50^2 = 2512.$$

(b) $\mathbb{E}[3X+2]$
- **Solution:**
$$\mathbb{E}[3X+2] = 3\mathbb{E}[X] + \mathbb{E}[2] = 3 \cdot 50 + 2 = 152.$$

(c) $\mathbb{E}\left[(X+2)^2\right]$
- **Solution:**
$$\mathbb{E}\left[(X+2)^2\right] = \mathbb{E}\left[X^2\right] + 4\mathbb{E}[X] + 4 = 2512 + 4 \cdot 50 + 4 = 2716.$$

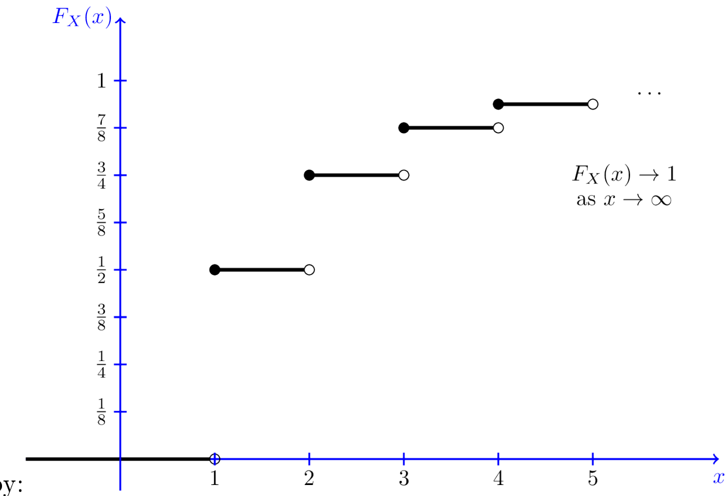
(d) $\mathrm{Var}[-X]$
- **Solution:**
$$\mathrm{Var}[-X] = (-1)^2 \mathrm{Var}(X) = 12$$

(e) $SD(2X)$.
- **Solution:**
$$SD(2X) = \sqrt{\mathrm{Var}(2X)} = \sqrt{2^2 \mathrm{Var}(X)} = \sqrt{48} = 2\sqrt{12}.$$

(7) Does there exists a random variable X such that $\mathbb{E}[X] = 4$ and $\mathbb{E}\left[X^2\right] = 10$? Why or why not ? (Hint: Look at its variance)
- **Solution:** Using the hint let's compute the variance of this random variable which would be $\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = 10 - 4^2 = -6$. But we know a random variable cannot have a negative variance. Thus no such random variable exists.

(8) Let X be the total number of text messages a random college student at Big State University receives in a year. Suppose you are given that the pmf of X is of the form
$$p_X(n) = \frac{c}{2^n}, \text{for } n = 0, 1, 2, 3, \ldots.$$
What must the value of c have to be if p_X is indeed a pmf of X? (Hint: Recall from Calculus 2 that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ whenever $0 < x < 1$)
- **Solution:** Using the property of pmf we know that since $X \in \{0, 1, 2, 3, \ldots\}$ then
$$\sum_{n=0}^{\infty} p_X(n) = 1,$$
so that
$$\sum_{n=0}^{\infty} \frac{c}{2^n} = 1.$$
Using the hint we can write the left hand side as
$$\sum_{n=0}^{\infty} \frac{c}{2^n} = c\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{c}{1-\frac{1}{2}} = 2c$$
thus $2c = 1$ which means
$$c = \frac{1}{2}.$$

(9) Let X be a discrete random variable with range given by $X \in \{1, 2, 3, \ldots\}$. Suppose the pmf of X is given by
$$p_X(n) = \frac{1}{2^n} \text{ for } n = 1, 2, 3, \ldots.$$
(a) Give a sketch of the graph of the CDF of X.



- **Solution:** The graph is give by:

(b) What is the value of $F_X(4)$.

- **Solution:** We have that

$$F_X(4) = \mathbb{P}(X \leq 4) = p(1) + p(2) + p(3) + p(4)$$
$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{15}{16}.$$

(c) Use the CDF to find $\mathbb{P}(X > 4)$.

- **Solution:** Using Part (b), we have that

$$\mathbb{P}(X > 4) = 1 - \mathbb{P}(X \leq 4) = 1 - F_X(4) = 1 - \frac{15}{16} = \frac{1}{16}.$$

(10) Suppose the CDF of X is given by

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3} & -1 \leq x < 1, \\ \frac{x}{3} & 1 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

(a) Plot this CDF.

(b) Find $\mathbb{P}(X > 1)$.

- **Solution:** We have

$$\mathbb{P}(X > 1) = 1 - \mathbb{P}(X \leq 1)$$
$$= 1 - F_X(1)$$
$$= 1 - \frac{1}{3} = \frac{2}{3}.$$

(c) Find $\mathbb{P}(X = 1)$.

- **Solution:** Once you draw the graph, it makes things much easier to see, (or use the proposition from Section 5.4), so

$$\begin{aligned}\mathbb{P}(X=1) &= \mathbb{P}(X \leq 1) - \mathbb{P}(X < 1) \\ &= F_X(1) - \lim_{x \to 1^-} F_X(x) \\ &= \frac{1}{3} - \frac{1}{3} = 0.\end{aligned}$$

(d) Find $\mathbb{P}(X=2)$.

- **Solution:** Once you draw the graph, it makes things much easier to see, so

$$\begin{aligned}\mathbb{P}(X=2) &= \mathbb{P}(X \leq 2) - \mathbb{P}(X < 2) \\ &= F_X(2) - \lim_{x \to 2^-} F_X(x) \\ &= 1 - \frac{2}{3} = \frac{1}{3}.\end{aligned}$$

CHAPTER 6

Some Discrete Distributions

(1) A UConn student claims that she can tell Dairy Bar ice cream from Friendly's ice cream. As a test, she is given ten samples of ice cream (each sample is either from the Dairy Bar or friendly's) and asked to identify each one. She is right eight times. What is the probability that she would be right exactly eight times if she guessed randomly for each sample?

- **Solution:** This should be modeled using a binomial random variable X, since there is a sequence of trials with the same probability of success in each one. If she guesses randomly for each sample, the probability that she will be right each time is $\frac{1}{2}$. Therefore,

$$\mathbb{P}(X=8)=\binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2=\frac{45}{2^{10}}.$$

(2) A Pharmaceutical company conducted a study on a new drug that is supposed to treat patients suffering from a certain disease. The study concluded that the drug did not help 25% of those who participated in the study. What is the probability that of 6 randomly selected patients, 4 will recover?

- **Solution:** $\binom{6}{4}(.75)^4(.25)^2$

(3) 20% of all students are left-handed. A class of size 20 meets in a room with 18 right-handed desks and 5 left-handed desks. What is the probability that every student will have a suitable desk?

- **Solution:** For each student to have the kind of desk he or she prefers, there must be no more than 18 right-handed students and no more than 5 left-handed students, so the number of left-handed students must be between 2 and 5 (inclusive). This means that we want the probability that there will be 2, 3, 4, or 5 left-handed students. We use the binomial distribution and get

$$\sum_{i=2}^{5}\binom{20}{i}\left(\frac{1}{5}\right)^i\left(\frac{4}{5}\right)^{20-i}.$$

(4) A ball is drawn from an urn containing 4 blue and 5 red balls. After the ball is drawn, it is replaced and another ball is drawn. Suppose this process is done 7 times.

(a) What is the probability that exactly 2 red balls were drawn in the 7 draws?

- **Solution:** $\binom{7}{2}\left(\frac{5}{9}\right)^2\left(\frac{4}{9}\right)^5$

(b) What is the probability that at least 3 blue balls were drawn in the 7 draws?

- **Solution:** $\mathbb{P}(X\geq 3)=1-\mathbb{P}(X\leq 2)=1-\binom{7}{0}\left(\frac{4}{9}\right)^0\left(\frac{5}{9}\right)^7-\binom{7}{1}\left(\frac{4}{9}\right)^1\left(\frac{5}{9}\right)^6-\binom{7}{2}\left(\frac{4}{9}\right)^2\left(\frac{5}{9}\right)^5$

(5) The expected number of typos on a page of the new Harry Potter book is .2. What is the probability that the next page you read contains
  (a) 0 typos ?
    - **Solution:** $e^{-.2}$
  (b) 2 or more typos?
    - **Solution:** $1 - e^{-.2} - .2e^{-.2} = 1 - 1.2e^{-.2}$.
  (c) Explain what assumptions you used.
    - **Solution:** Since each word has a small probability of being a typo, the number of typos should be approximately be Poisson distributed.

(6) The monthly average number of car crashes in Storrs, CT is 3.5. What is the probability that there will be
  (a) at least 2 accidents in the next month?
    - **Solution:** $1 - e^{-3.5} - 3.5e^{-3.5} = 1 - 4.5e^{-3.5}$
  (b) at most 1 accident in the next month?
    - **Solution:** $4.5e^{-3.5}$
  (c) Explain what assumptions you used.
    - **Solution:** Since each accident has a small probability it seems reasonable to suppose that the number of car accidents is approximately Poisson distributed.

(7) Suppose that the average number of burglaries in New York City in a week is 2.2. Approximate the probability that there will be
  (a) no burglaries in the next week;
    - **Solution:** $e^{-2.2}$
  (b) at least 2 burglaries in the next week.
    - **Solution:** $1 - e^{-2.2} - 2.2e^{-2.2} = 1 - 3.2e^{-2.2}$.

(8) The number of accidents per working week in a particular shipyard is Poisson distributed with mean 0.5. Find the probability that:
  (a) In a particular week there will be at least 2 accidents.
    - **Solution:** We have $\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X \leq 1) = 1 - e^{.5}\frac{(.5)^0}{0!} - e^{.5}\frac{(.5)^1}{1!}$.
  (b) In a particular two week period there will be exactly 5 accidents.
    - **Solution:** In two weeks the average number of accidents will be $\lambda = .5 + .5 = 1$. Then $\mathbb{P}(X = 5) = e^{-1}\frac{1^5}{5!}$.
  (c) In a particular month (i.e. 4 week period) there will be exactly 2 accidents.
    - **Solution:** In a 4 week period the average number of accidents will be $\lambda = 4 \cdot (.5) = 2$. Then $\mathbb{P}(X = 2) = e^{-2}\frac{2^2}{2!}$.

(9) Jennifer is baking cookies. She mixes 400 raisins and 600 chocolate chips into her cookie dough and ends up with 500 cookies.
  (a) Find the probability that a randomly picked cookie will have three raisins in it.
    - **Solution:** This calls for a Poisson random variable R. The average number of raisins per cookie is .8, so we take this as our λ . We are asking for $\mathbb{P}(R = 3)$, which is $e^{-.8}\frac{(.8)\&3}{3!} \approx .0383$.
  (b) Find the probability that a randomly picked cookie will have at least one chocolate chip in it.
    - **Solution:** This calls for a Poisson random variable C. The average number of chocolate chips per cookie is 1.2, so we take this as our λ. We are asking for $\mathbb{P}(C \geq 1)$, which is $1 - \mathbb{P}(C = 0) = 1 - e^{-1.2}\frac{(1.2)^0}{0!} \approx .6988$.

(c) Find the probability that a randomly picked cookie will have no more than two bits in it (a bit is either a raisin or a chocolate chip).

- **Solution:** This calls for a Poisson random variable B. The average number of bits per cookie is $.8 + 1.2 = 2$, so we take this as our λ. We are asking for $\mathbb{P}(B \leq 2)$, which is $\mathbb{P}(B = 0) + \mathbb{P}(B = 1) + \mathbb{P}(B = 2) = e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!} \approx .6767$.

(10) Chevy has three factories that produces their car called *Camaro*. The average number of factory defects per Camaro is2.2 when built by the first factory, 4 when built by the second factory and 1.5 when built by the third factory. Suppose you buy a brand new Camaro from your local Chevy dealer. If your Camaro is equally likely to be typed by either factories, approximate the probability that it will have no defects. Assume factory defects per Camaro is Poisson distributed. (Hint: Law of total probability)

- **Solution:** Let us partition the sample space S by $S = F_1 \cup F_2 \cup F_3$ where F_1 is the event that a Camaro is built by Factory 1, F_2 is the event that a Camaro is built by Factory 2, and F_3 is the event that a Camaro is built by Factory 3. Then

$$\mathbb{P}(F_1) = \frac{1}{3}, \ \mathbb{P}(F_2) = \frac{1}{3} \ \mathbb{P}(F_3) = \frac{1}{3}.$$

Let N be the event that your camaro has no defects. Then by the law of total probability

$$\begin{aligned}
\mathbb{P}(N) &= \mathbb{P}(N \mid F_1)\,\mathbb{P}(F_1) + \mathbb{P}(N \mid F_2)\,\mathbb{P}(F_2) + \mathbb{P}(N \mid F_2)\,\mathbb{P}(F_2) \\
&= \left(e^{-2.2}\frac{(2.2)^0}{0!}\right)\cdot\frac{1}{3} + \left(e^{-4}\frac{(4)^0}{0!}\right)\cdot\frac{1}{3} + \left(e^{-1.5}\frac{(1.5)^0}{0!}\right)\cdot\frac{1}{3} \\
&= \frac{1}{3}\left(e^{-2.2} + e^{-4} + e^{-1.5}\right).
\end{aligned}$$

(11) A roulette wheel has 38 numbers on it: the numbers 0 through 36 and a 00. Suppose that Lauren always bets that the outcome will be a number between 1 and 18 (including 1 and 18).

(a) What is the probability that Lauren will lose her first 6 bets.

- **Solution:** $\left(1 - \frac{18}{38}\right)^6$.

(b) What is the probability that Lauren will first win on her sixth bet?

- **Solution:** $\left(1 - \frac{18}{38}\right)^5 \frac{18}{38}$.

(c) What is the expected number of bets until her first win?

- **Solution:** Since $X \sim \text{Geometric}\left(\frac{18}{38}\right)$ with $p = \frac{18}{38}$ then

$$\mathbb{E}X = \frac{1}{p} = \frac{38}{18} \approx 2.111\bar{1}.$$

CHAPTER 7

Continuous distributions

(1) Let X be a random variable with probability density function

$$f(x)=\begin{cases} cx\,(5-x) & 0\le x\le 5\\ 0 & \text{otherwise}\end{cases}.$$

(a) What is the value of c?

- **Solution:** We must have that $\int_{-\infty}^{\infty} f(x)dx = 1$, thus

$$1=\int_0^5 cx(5-x)dx=\left[c\left(\frac{5x^2}{2}-\frac{x^3}{3}\right)\right]_0^5$$

and so we must have that $c=6/125$.

(b) What is the cumulative distribution function of X? That is, find $F_X(x)=\mathbb{P}(X\le x)$.

- **Solution:** We have that

$$\begin{aligned} F_X(x) &= \mathbb{P}(X\le x)=\int_{-\infty}^{x} f(y)dy\\ &= \int_0^x \frac{6}{125}y\,(5-y)\,dx=\frac{6}{125}\left[\left(\frac{5y^2}{2}-\frac{y^3}{3}\right)\right]_0^x\\ &= \frac{6}{125}\left(\frac{5x^2}{2}-\frac{x^3}{3}\right).\end{aligned}$$

(c) Use your answer in part (b) to find $\mathbb{P}(2\le X\le 3)$.

- **Solution:** We have

$$\begin{aligned}\mathbb{P}(2\le X\le 3) &= \mathbb{P}(X\le 3)-\mathbb{P}(X<2)\\ &= \frac{6}{125}\left(\frac{5\cdot 3^2}{2}-\frac{3^3}{3}\right)-\frac{6}{125}\left(\frac{5\cdot 2^2}{2}-\frac{2^3}{3}\right)\\ &= .296.\end{aligned}$$

(d) What is $\mathbb{E}[X]$?

- **Solution:** We have

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} xf_X(x)dx=\int_0^5 x\cdot\frac{6}{125}x(5-x)dx\\ &= 2.5.\end{aligned}$$

(e) What is $\mathrm{Var}(X)$?

- **Solution:** We need to first compute

$$\begin{aligned}\mathbb{E}\left[X^2\right] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^5 x^2 \cdot \frac{6}{125}x(5-x)dx \\ &= 7.5.\end{aligned}$$

Then

$$\operatorname{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = 7.5 - (2.5)^2 = 1.25.$$

(2) UNH students have designed the new u-phone. They have determined that the lifetime of a U-Phone is given by the random variable X (measured in hours), with probability density function

$$f(x) = \begin{cases} \frac{10}{x^2} & x \geq 10 \\ 0 & x \leq 10 \end{cases}.$$

(a) Find the probability that the u-phone will last more than 20 hours?
- **Solution:** We have

$$\int_{20}^{\infty} \frac{10}{x^2}dx = \frac{1}{2}.$$

(b) What is the cumulative distribution function of X? That is, find $F_X(x) = \mathbb{P}(X \leq x)$.
- **Solution:** We have

$$F(x) = \mathbb{P}(X \leq x) = \int_{10}^{x} \frac{10}{y^2}dy = 1 - \frac{10}{x}$$

for $x > 10$, and $F(x) = 0$ for $x < 10$.

(c) Use part (b) to help you find $\mathbb{P}(X \geq 35)$?
- **Solution:** We have

$$\begin{aligned}\mathbb{P}(X \geq 35) &= 1 - \mathbb{P}(X < 35) = 1 - F_X(35) \\ &= 1 - \left(1 - \frac{10}{35}\right) = \frac{10}{35}.\end{aligned}$$

(3) Suppose the random variable X has a density function

$$f(x) = \begin{cases} \frac{2}{x^2} & x > 2, \\ 0 & x \leq 2. \end{cases}$$

Compute $\mathbb{E}[X]$.
- **Solution:** Since $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx$ then

$$\mathbb{E}[X] = \int_2^{\infty} \frac{2}{x}dx = \infty.$$

(4) An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} \frac{3}{x^4} & x > 1, \\ 0 & \text{, otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, calculate the probability that it is insured for less than 2.

- **Solution:** Note that the distribution function is given by

$$F(x) = \mathbb{P}(X \le x) = \int_1^x 3y^{-4}dy = 1 - \frac{1}{x^3}.$$

Then

$$\begin{aligned}
\mathbb{P}(X < 2 \mid X \ge 1.5) &= \frac{\mathbb{P}((X<2) \text{ and } (X \ge 1.5))}{\mathbb{P}(X \ge 1.5)} = \frac{\mathbb{P}(X<2) - \mathbb{P}(X<1.5)}{\mathbb{P}(X \ge 1.5)} \\
&= \frac{F(2) - F(1.5)}{1 - F(1.5)} = \frac{\left(1 - 2^{-3}\right) - \left(1 - 1.5^{-3}\right)}{1 - \left(1 - 1.5^{-3}\right)} \\
&= \frac{37}{64} = .578.
\end{aligned}$$

(5) The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = \frac{7}{10}$, find the values of a and b.

- **Solution:** We need to use the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\mathbb{E}[X] = \frac{7}{10}$. The first one gives us,

$$1 = \int_0^1 \left(a + bx^2\right) dx = a + \frac{b}{3}$$

and the second one give us

$$\frac{7}{10} = \int_0^1 x\left(a + bx^2\right) dx = \frac{a}{2} + \frac{b}{4}.$$

Solving these equations gives us

$$a = \frac{1}{5}, \text{ and } b = \frac{12}{5}.$$

(6) Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{1}{a-1} & 1 < x < a \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\mathbb{E}[X] = 6\mathrm{Var}(X)$. Find the value of a.

- **Solution:** Note that

$$\mathbb{E}X = \int_1^a \frac{x}{a-1}dx = \frac{1}{2}a + \frac{1}{2}.$$

Also

$$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

then we need

$$\mathbb{E}X^2 = \int_1^a \frac{x^2}{a-1}dx = \frac{1}{3}a^2 + \frac{1}{3}a + \frac{1}{3}.$$

Then

$$\begin{aligned} Var(X) &= \left(\frac{1}{3}a^2 + \frac{1}{3}a + \frac{1}{3}\right) - \left(\frac{1}{2}a + \frac{1}{2}\right)^2 \\ &= \frac{1}{12}a^2 - \frac{1}{6}a + \frac{1}{12}. \end{aligned}$$

Then using $\mathbb{E}[X] = 6\mathrm{Var}(X)$ we solve and get $\frac{1}{2}a^2 - \frac{3}{2}a = 0$ which we get $a = 3$.

(7) Suppose you order a pizza from your favorite pizzaria at 7:00 pm, knowing that the time it takes for your pizza to be ready is uniformly distributed between 7:00 pm and 7:30 pm.

(a) What is the probability that you will have to wait longer than 10 minutes for your pizza?

- **Solution:** Note that X is uniformly distributed over $(0, 30)$. Then

$$\mathbb{P}(X > 10) = \frac{2}{3}.$$

(b) If at 7:15pm, the pizza has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

- **Solution:** Note that X is uniformly distributed over $(0, 30)$. Then

$$\mathbb{P}(X > 25 \mid X > 15) = \frac{\mathbb{P}(X > 25)}{\mathbb{P}(X > 15)} = \frac{5/30}{15/30} = 1/3.$$

(8) Suppose an insurance policy offers Toy Insurance. The rules of the **benefit paid** under this policy are the following: it reimburses a loss up to a **benefit** limit of only 10 dollars, otherwise it pays full amount of the loss. The insurance company calculates that the policyholder's toy **loss**, Y, follows a distribution with density function given by:

$$f_Y(y) = \begin{cases} \frac{2}{y^3} & \text{for } y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let X be the **benefit paid** under this insurance policy. What is the expected value of X? (The question is **NOT** asking for the expected value of the loss Y)

- **Solution:** The question is asking for the expected **benefit** X the Insurance company expects to pay under this policy, which is based on the loss. In fact, the problem says that as long as the loss is less than or equal to 10, it pays the exact amount of the loss, but if the loss is greater than 10, then it caps the benefit at 10. In symbols, this says

$$X = \begin{cases} Y & Y \leq 10 \\ 10 & Y > 10 \end{cases}.$$

Hence using the definition of expected value,

$$
\begin{aligned}
\mathbb{E}X &= \mathbb{E}\left[Y; Y \leq 10\right] + \mathbb{E}\left[10; Y > 10\right] \\
&= \int_{-\infty}^{10} y f_Y(y) dy + \int_{10}^{\infty} 10 f_Y(y) dy \\
&= \int_{1}^{10} y \cdot \frac{2}{y^3} dy + \int_{10}^{\infty} 10 \cdot \frac{2}{y^3} dy \\
&= 2\int_{1}^{10} \frac{1}{y^2} dy + 20\int_{10}^{\infty} \frac{1}{y^3} dy \\
&= 2\left[-\frac{1}{y}\right]_1^{10} + 20\left[-\frac{1}{2y^2}\right]_{10}^{\infty} \\
&= -2\left[\frac{1}{10} - 1\right] - 10\left[\frac{1}{\infty} - \frac{1}{100}\right] \\
&= -\frac{2}{10} + 2 + \frac{1}{10} \\
&= \frac{19}{10} \\
&= 1.9.
\end{aligned}
$$

CHAPTER 8

Normal Distributions

(1) Suppose X is a normally distributed random variable with $\mu = 10$ and $\sigma^2 = 36$. Find

(a) $\mathbb{P}(X > 5)$,

- **Solution:**

$$\begin{aligned}\mathbb{P}(X>5) &= \mathbb{P}\left(Z > \frac{5-10}{6}\right) = \mathbb{P}(Z > -.8333)\\ &= 1-\mathbb{P}(Z \leq -.8333) = 1-\Phi(-.8333)\\ &= 1-(1-\Phi(.8333)) = .7977\end{aligned}$$

(b) $\mathbb{P}(4 < X < 16)$,

- **Solution:** $2\Phi(1) - 1 = .6827$

(c) $\mathbb{P}(X < 8)$.

- **Solution:** $1 - \Phi(.3333) = .3695$**.**

(2) The height of maple trees at age 10 are estimated to be normally distributed with mean 200 cm and variance 64 cm. What is the probability a maple tree at age 10 grows more than 210cm?

- **Solution:** We have $\mu = 200$ and $\sigma = \sqrt{64} = 8$. Then

$$\begin{aligned}\mathbb{P}(X>210) &= \mathbb{P}\left(Z > \frac{210-200}{8}\right) = \mathbb{P}(Z > 1.25)\\ &= 1-\Phi(1.25) = .1056.\end{aligned}$$

(3) The peak temperature T, in degrees Fahrenheit, on a July day in Antarctica is a Normal random variable with a variance of 225. With probability .5, the temperature T exceeds 10 degrees.

(a) What is $\mathbb{P}(T > 32)$, the probability the temperature is above freezing?

- **Solution:** We have $\sigma = \sqrt{225} = 15$. Since $\mathbb{P}(X > 10) = .5$ then we must have that $\mu = 10$ since the pdf of the normal distribution is symmetric. Then

$$\begin{aligned}\mathbb{P}(T>32) &= \mathbb{P}\left(Z > \frac{32-10}{15}\right)\\ &= 1-\Phi(1.47) = .0708.\end{aligned}$$

(b) What is $\mathbb{P}(T < 0)$?

- **Solution:** We have $\mathbb{P}(T < 0) = \Phi(-.67) = 1 - \Phi(.67) = .2514$.

(4) The salaries of UConn professors is approximately normally distributed. Suppose you know that 33 percent of professors earn less than \$80,000. Also 33 percent earn more than \$120,000.

(a) What is the probability that a UConn professor makes more than \$100,000?

- **Solution:** First we need to figure out what μ and $\sigma$are. Note that

$$\begin{aligned}\mathbb{P}(X \leq 80{,}000) = .33 \quad &\iff \quad \mathbb{P}\left(Z < \frac{80{,}000 - \mu}{\sigma}\right) = .33\\ &\iff \quad \Phi\left(\frac{80{,}000 - \mu}{\sigma}\right) = .33\end{aligned}$$

and since $\Phi(.44) = .67$ then $\Phi(-.44) = .33$. Then we must have

$$\frac{80{,}000 - \mu}{\sigma} = -.44.$$

Similarly, since

$$\begin{aligned}\mathbb{P}(X > 120{,}000) = .33 \quad &\iff \quad 1 - \mathbb{P}(X \leq 120{,}000) = .33\\ &\iff \quad 1 - \Phi\left(\frac{120{,}000 - \mu}{\sigma}\right) = .33\\ &\iff \quad \Phi\left(\frac{120{,}000 - \mu}{\sigma}\right) = .67\end{aligned}$$

Now again since $\Phi(.44) = .67$ then

$$\frac{120{,}000 - \mu}{\sigma} = .44.$$

Solving the equations

$$\frac{80{,}000 - \mu}{\sigma} = -.44 \text{ and } \frac{120{,}000 - \mu}{\sigma} = .44,$$

simultaneously we have that

$$\mu = 100{,}000 \text{ and } \sigma \approx 45{,}454.5.$$

Then

$$\mathbb{P}(X > 100{,}000) = .5.$$

(b) What is the probability that a UConn professor makes between \$70,000 and \$80,000?

- **Solution:** We have

$$\mathbb{P}(70{,}000 < X < 80{,}000) \approx .0753.$$

(5) Suppose X is a normal random variable with mean 5. If $\mathbb{P}(X > 0) = .8888$, approximately what is $Var(X)$?

- **Solution:** Since $\mathbb{P}(X > 0) = .8888$, then

$$\begin{aligned}\mathbb{P}(X > 0) = .8888 \quad &\iff \quad \mathbb{P}\left(Z > \frac{0-5}{\sigma}\right) = .8888\\ &\iff \quad 1 - \mathbb{P}\left(Z \leq -\frac{5}{\sigma}\right) = .8888\\ &\iff \quad 1 - \Phi\left(-\frac{5}{\sigma}\right) = .8888\\ &\iff \quad 1 - \left(1 - \Phi\left(\frac{5}{\sigma}\right)\right) = .8888\\ &\iff \quad \Phi\left(\frac{5}{\sigma}\right) = .8888.\end{aligned}$$

Using the table we see that $\Phi(1.22) = .8888$, thus we must have that

$$\frac{5}{\sigma} = 1.22$$

and solving this gets us $\sigma = 4.098$, hence $\sigma^2 \approx 16.8$.

(6) The shoe size of a UConn basketball player is normally distributed with mean 12 inches and variance 4 inches. Ten percent of all UConn basketball players have a shoe size greater than c inches. Find the value of c.

- **Solution:** Note that

$$\begin{aligned}\mathbb{P}(X > c) = .10 \quad &\Longleftrightarrow \quad \mathbb{P}\left(Z > \frac{c-12}{2}\right) = .10 \\ &\Longleftrightarrow \quad 1 - \mathbb{P}\left(Z \leq \frac{c-12}{2}\right) = .10 \\ &\Longleftrightarrow \quad \mathbb{P}\left(Z \leq \frac{c-12}{2}\right) = .9 \\ &\Longleftrightarrow \quad \Phi\left(\frac{c-12}{2}\right) = .9\end{aligned}$$

Using the table we see that $\Phi(1.28) = .90$, thus we must have that

$$\frac{c-12}{2} = 1.28$$

and solving this gets us $c = 14.56$.

CHAPTER 9

Normal Approximation to the binomial

(1) Suppose that we roll 2 dice 180 times. Let E be the event that we roll two fives no more than once.

(a) Find the exact probability of E.

- **Solution:** The probability of rolling two ves in a particular roll is $\frac{1}{36}$, so the probability that we roll two fives no more than once in 180 rolls is

$$p = \binom{180}{0}\left(\frac{35}{36}\right)^{180} + \binom{180}{1}\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)^{179} \approx .0386.$$

(b) Approximate $\mathbb{P}(E)$ using the normal distribution.

- **Solution:** We want the number of successes to be 0 or 1, so we want $\mathbb{P}(0 \leq S_{180} \leq 1)$. Since the binomial is integer-valued, we apply the continuity correction and calculate $\mathbb{P}(-.5 \leq S_{180} \leq 1.5)$instead. We calculate that the expected value is $\mu = 180 \cdot p = 5$ and the standard deviation $\sigma = \sqrt{180p(1-p)} \approx 2.205$. Now, as always, we convert this question to a question about the standard normal random variable Z,

$$\begin{aligned} \mathbb{P}(-.5 \leq S_{180} \leq 1.5) &= \mathbb{P}\left(\frac{.5-5}{2.205} \leq Z \leq \frac{1.5-5}{2.205}\right) = \mathbb{P}(-2.49 < Z < -1.59) \\ &= (1-\Phi(1.59)) - (1-\Phi(2.49)) \\ &= (1-.9441) - (1-.9936) = .0495. \end{aligned}$$

(c) Approximate $\mathbb{P}(E)$ using the Poisson distribution.

- **Solution:** We use $\lambda = np = 5$ (note that we calculated this already in (b)!). Now we see that

$$\mathbb{P}(E) \approx e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!} \approx .0404.$$

CHAPTER 10

Some continuous distributions

(1) Suppose that the time required to replace a car's windshield can be represented by an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$.

(a) What is the probability that it will take at least 3 hours to replace a windshield?

- **Solution:** We have

$$\begin{aligned}\mathbb{P}(X > 3) &= 1 - \mathbb{P}(0 < X < 3) \\ &= 1 - \int_0^3 \frac{1}{2} e^{-\frac{x}{2}} dx \\ &= e^{-\frac{3}{2}} \approx .2231.\end{aligned}$$

(b) What is the probability that it will take at least 5 hours to replace a windshield given that it hasn't been finished after 2 hours?

- **Solution:** There are two ways to do this. The longer one is to calculate $\mathbb{P}(X > 5 \mid X > 2)$. The shorter one is to remember that the exponential distribution is memoryless and to observe that $\mathbb{P}(X > t + 3 \mid X > t) = P(X > 3)$, so the answer is the same as the answer to (a).

(2) The number of years a u-phone functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If Pat buys a used u-phone, what is the probability that it will be working after an additional 8 years?

- **Solution:** e^{-1}

(3) Suppose that the time (in minutes) required to check out a book at the library can be represented by an exponentially distributed random variable with parameter $\lambda = \frac{2}{11}$.

(a) What is the probability that it will take at least 5 minutes to check out a book?

- **Solution:** Recall a formula $\mathbb{P}(X > a) = e^{-\lambda a}$, then

$$\mathbb{P}(X > 5) = e^{-\frac{10}{11}}.$$

(b) What is the probability that it will take at least 11 minutes to check out a book given that you've already waited for 6 minutes?

- **Solution:** We use the memoryless property

$$\begin{aligned}\mathbb{P}(X > 11 \mid X > 6) &= \mathbb{P}(X > 6 + 5 \mid X > 6) \\ &= \mathbb{P}(X > 5) = e^{-\frac{10}{11}}.\end{aligned}$$

(4) Let X be an exponential random variable with mean $\mathbb{E}[X] = 1$. Define a new random variable $Y = e^X$. Find the PDE of Y, $f_Y(y)$, and then use the PDF to compute the mean of Y.

- **Solution1:** Since $\mathbb{E}[X] = 1$ then we know that λ=1. Then it's pdf and cdf is

$$f_X(x) = \begin{cases} e^{-x} & , x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Step1: Find the cdf of Y and write in terms of F_X:
– By using the given relation,

$$\begin{aligned} F_y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}\left(e^X \leq y\right) \\ &= \mathbb{P}(X \leq \ln y) \\ &= F_X(\ln y) \end{aligned}$$

- **Step2:** Then use the relation $f_Y(y) = F_Y'(y)$ and take a derivative of both sides,

$$\begin{aligned} f_Y(y) &= F_Y'(y) \\ &= \frac{d}{dy}\left[F_X(\ln y)\right], \text{ use chain rule} \\ &= F_X'(\ln y) \cdot (\ln y)' \\ &= f_X(\ln y) \cdot \frac{1}{y}, \text{ since } F_X' = f_X \\ &= \begin{cases} e^{-\ln y} \cdot \frac{1}{y} & 0 < \ln y < \infty \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} e^{\ln y^{-1}} \cdot \frac{1}{y} & e^0 < y < e^{\infty} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{y^2} & 1 < y < \infty \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

- **Step3:** Now to compute $\mathbb{E}Y$ using f_Y we have

$$\begin{aligned} \mathbb{E}Y &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_1^{\infty} y \frac{1}{y^2} dy \\ &= \int_1^{\infty} \frac{1}{y} dy \\ &= [\ln y]_1^{\infty} \\ &= \lim_{y \to \infty} \ln y - \ln 1 \\ &= \infty. \end{aligned}$$

so $\mathbb{E}Y = \infty$.

- **Solution2:(plugging the CDF directly since it is known)** Since $\mathbb{E}[X] = 1$ then we know that λ=1. Then it's pdf and cdf is

$$\begin{aligned} f_X(x) &= e^{-x}, x \geq 0 \\ F_X(x) &= 1 - e^{-x}, \; x \geq 0. \end{aligned}$$

By using the given relation,

$$F_y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}\left(e^X \leq y\right) = \mathbb{P}(X \leq \ln y) = F_X(\ln y)$$

and so

$$F_Y(y) = 1 - e^{-\ln y} = 1 - \frac{1}{y}, \text{ when } \ln(y) \geq 0,$$

taking derivatives we get

$$f_Y(y) = \frac{dF_y(y)}{dy} = \frac{1}{y^2}, \text{ when } y \geq 1.$$

Thus

$$f_Y(y) = \begin{cases} \frac{1}{y^2} & y \geq 1 \\ 0 & y < 1 \end{cases}$$

and thus

$$\begin{aligned} \mathbb{E}Y &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_1^{\infty} y \frac{1}{y^2} dy \\ &= \int_1^{\infty} \frac{1}{y} dy \\ &= [\ln y]_1^{\infty} \\ &= \lim_{y \to \infty} \ln y - \ln 1 \\ &= \infty. \end{aligned}$$

(5) Suppose that X has an exponential distribution with parameter $\lambda = 1$. Let $c > 0$. Show that $Y = \frac{X}{c}$ is exponential with parameter $\lambda = c$.

- **Solution:** Since X is exponential with parameter 1, then it's pdf and cdf is

$$f_X(x) = \begin{cases} e^{-x} & , x \geq 0 \\ 0 & x < 0 \end{cases}.$$

- <u>**Step1:**</u> Find the cdf of Y and write in terms of F_X:
  - By using the given relation,

$$\begin{aligned} F_y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}\left(\frac{X}{c} \leq y\right) \\ &= \mathbb{P}(X \leq cy) \\ &= F_X(cy) \end{aligned}$$

- **Step2:** Then use the relation $f_Y(y) = F'_Y(y)$ and take a derivative of both sides,

$$\begin{aligned} f_Y(y) &= F'_Y(y) \\ &= \frac{d}{dy}\left[F_X(cy)\right], \text{ use chain rule} \\ &= F'_X(cy)\cdot(cy)' \\ &= f_X(cy)\cdot c, \text{ since } F'_X = f_X \\ &= \begin{cases} e^{-cy}\cdot c & 0 < cy < \infty \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} ce^{-cy} & 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

  – Note that since

$$f_Y(y) = \begin{cases} ce^{-cy}, & y \geq 0 \\ 0 & y < 0 \end{cases}$$

  then this is the pdf of an exponential with parameter $\lambda = c$.

(6) Let X be a uniform random variable over $(0,1)$. Define a new random variable $Y = e^X$. Find the probability density function of Y, $f_Y(y)$.

- **Solution1:** Since X is uniform over $(0,1)$, then it's pdf and cdf are

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Step1: Find the cdf of Y and write in terms of F_X:

  – By using the given relation,

$$\begin{aligned} F_y(y) &= \mathbb{P}\left(e^X \leq y\right) \\ &= \mathbb{P}\left(X \leq \ln y\right) \\ &= F_X\left(\ln y\right) \end{aligned}$$

- **Step2:** Then use the relation $f_Y(y) = F'_Y(y)$ and take a derivative of both sides,

$$\begin{aligned} f_Y(y) &= F'_Y(y) \\ &= \frac{d}{dy}\left[F_X(\ln y)\right], \text{ use chain rule} \\ &= F'_X(\ln y)\cdot(\ln y)' \\ &= f_X(\ln y)\cdot\frac{1}{y}, \text{ since } F'_X = f_X \\ &= \begin{cases} 1\cdot\frac{1}{y} & 0 < \ln y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

- **Solution2:(pugging the CDF directly)** Since X is uniform over $(0,1)$, then it's pdf and cdf are

$$\begin{aligned} f_X(x) &= 1 \quad , \quad 0 \le x < 1 \\ F_X(x) &= x \quad , \quad 0 \le x < 1. \end{aligned}$$

By using the given relation,

$$F_y(y) = \mathbb{P}(Y \le y) = \mathbb{P}\left(e^X \le y\right) = \mathbb{P}(X \le \ln y) = F_X(\ln y)$$

and so

$$F_Y(y) = \ln y, \text{ when } 0 \le \ln y < 1.$$

taking derivatives we get

$$f_Y(y) = \frac{dF_y(y)}{dy} = \frac{1}{y}, \text{ when } 1 < y < e^1.$$

so that

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}.$$

(7) Suppose an amazon box always has a square base with height twice as much as the length of its base. Suppose it is known that the side length of the square base is given by a random variable X in inches with PDF given by

$$f_X(x) = \begin{cases} \frac{1}{9}x^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}.$$

Find the PDF of Y, the volume of the box and use the PDF to calculate the mean volume of an Amazon box.

- **Solution:**
- The volume of an Amazon box is given by

$$\begin{aligned} Y &= \text{length} \times \text{weidth} \times \text{height} \\ &= X \cdot X \cdot (2X) \\ &= 2X^3. \end{aligned}$$

- **Step1:** Find the cdf of Y and write in terms of F_X:
  - By using the given relation,

$$\begin{aligned} F_y(y) &= \mathbb{P}\left(2X^3 \le y\right) \\ &= \mathbb{P}\left(X^3 \le \frac{y}{2}\right) \\ &= \mathbb{P}\left(X \le \frac{y^{1/3}}{2^{1/3}}\right) \\ &= F_X\left(\frac{y^{1/3}}{2^{1/3}}\right) \end{aligned}$$

- **Step2:** Then use the relation $f_Y(y) = F_Y'(y)$ and take a derivative of both sides,

$$
\begin{aligned}
f_Y(y) &= F_Y'(y) \\
&= \frac{d}{dy}\left[F_X\left(\frac{y^{1/3}}{2^{1/3}}\right)\right], \text{ use chain rule} \\
&= F_X'\left(\frac{y^{1/3}}{2^{1/3}}\right)\cdot\left(\frac{y^{1/3}}{2^{1/3}}\right)' \\
&= f_X\left(\frac{y^{1/3}}{2^{1/3}}\right)\cdot\frac{1}{3\cdot 2^{1/3}\cdot y^{2/3}}, \text{ since } F_X' = f_X \\
&= \begin{cases} \frac{1}{9}\frac{y^{2/3}}{2^{2/3}}\cdot\frac{1}{3\cdot 2^{1/3}\cdot y^{2/3}} & 0 < \frac{y^{1/3}}{2^{1/3}} < 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{54} & 0 < y^{1/3} < 3\cdot 2^{1/3} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{54} & 0 < y < 3^3\cdot 2 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{54} & 0 < y < 54 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

this shows that $Y \sim U[0, 54]$ hence its mean is

$$\mathbb{E}Y = \frac{54}{2} = 27$$

CHAPTER 11

Multivariate distributions

(1) Suppose that 2 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X equal 1 if the first ball selected is white and zero otherwise. Let Y equal 1 if the second ball selected is white and zero otherwise. Find the probability mass function of X, Y.

- **Solution:** We have

$$\begin{aligned} p(0,0) &= \mathbb{P}(X=0, Y=0) = \mathbb{P}(RR) = \frac{8 \cdot 7}{13 \cdot 12} = \frac{14}{39}, \\ p(1.0) &= \mathbb{P}(X=1, Y=0) = \mathbb{P}(WR) = \frac{5 \cdot 8}{13 \cdot 12} = \frac{10}{39}, \\ p(0,1) &= \mathbb{P}(X=0, Y=1) = \mathbb{P}(RW) = \frac{8 \cdot 5}{13 \cdot 12} = \frac{10}{39}, \\ p(1,1) &= \mathbb{P}(X=1, Y=1) = \mathbb{P}(WW) = \frac{5 \cdot 4}{13 \cdot 12} = \frac{5}{39}. \end{aligned}$$

(2) Suppose you roll two fair dice. Find the probability mass function of X and Y, where X is the largest value obtained on any die, and Y is the sum of the values.

- **Solution:** First we need to figure what values X, Y can attain. Note that X can be any of $1, 2, 3, 4, 5, 6$, But Y is the sum, can only be as low as 2 and as high as 12. First we make a table of possibilities for (X, Y) given the values of the die. Recall X is the largest of the two, and Y is the sum of them. The possible outcomes are given by:
-

1st Die\2nd die	1	2	3	4	5	6
1	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,6)$	$(6,7)$
2	$(2,3)$	$(2,4)$	$(3,5)$	$(4,6)$	$(5,7)$	$(6,8)$
3	$(3,4)$	$(3,5)$	$(3,6)$	$(4,7)$	$(5,8)$	$(6,9)$
4	$(4,5)$	$(4,6)$	$(4,7)$	$(4,8)$	$(5,9)$	$(6,10)$
5	$(5,6)$	$(5,7)$	$(5,8)$	$(5,9)$	$(5,10)$	$(6,11)$
6	$(6,7)$	$(6,8)$	$(6,9)$	$(6,10)$	$(6,11)$	$(6,12)$

- Then we make a table of the pmf $p(x, y)$.

$X \backslash Y$	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{1}{36}$	0	0	0	0	0	0	0	0	0	0
2	0	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	0	0
3	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0
4	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0
5	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0
6	0	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(3) Suppose the joint density function of X and Y is $f(x, y) = \frac{1}{4}$ for $0 < x < 2$ and $0 < y < 2$.

(a) Calculate $\mathbb{P}\left(\frac{1}{2} < X < 1, \frac{2}{3} < Y < \frac{4}{3}\right)$.

- **Solution:** We integrate the pdf over the bounds and get

$$\int_{\frac{1}{2}}^{1}\int_{\frac{2}{3}}^{\frac{4}{3}}\frac{1}{4}dydx=\frac{1}{4}\left(1-\frac{1}{2}\right)\left(\frac{4}{3}-\frac{2}{3}\right)=\frac{1}{12}.$$

(b) Calculate $\mathbb{P}(XY<2)$.

- **Solution:** We need to find the region that is within $0<x,y<2$ and $y<\frac{2}{x}$. (Draw a picture!!! Please!) We get two regions from this. One with bounds $0<x<1, 0<y<2$ and the other region being $1<x<2, 0<y<\frac{2}{x}$. Then

$$\begin{aligned}\mathbb{P}(XY<2) &= \int_0^1\int_0^2\frac{1}{4}dydx+\int_1^2\int_0^{\frac{2}{x}}\frac{1}{4}dydx\\ &= \frac{1}{2}+\int_1^2\frac{1}{2x}dx\\ &= \frac{1}{2}+\frac{\ln 2}{2}.\end{aligned}$$

(c) Calculate the marginal distributions $f_X(x)$ and $f_Y(y)$.

- **Solution:** Recall that

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)dy=\int_0^2\frac{1}{4}dy=\frac{1}{2}$$

for $0<x<2$ and 0 otherwise. By symmetry, f_Y is equal to the same thing.

(4) The joint probability density function of X and Y is given by

$$f(x,y)=e^{-(x+y)},\ \ 0\le x<\infty, 0\le y<\infty.$$

Find $\mathbb{P}(X<Y)$.

- **Solution:** Draw a picture of the region and note the bounds are

$$\begin{aligned}\mathbb{P}(X<Y) &= \int_0^{\infty}\int_0^{y}e^{-(x+y)}dxdy=\int_0^{\infty}\left[-e^{-2y}+e^{-y}\right]dy\\ &= \left[\frac{1}{2}e^{-2y}-e^{-y}\right]_0^{\infty}=0-\left(\frac{1}{2}-1\right)=\frac{1}{2}.\end{aligned}$$

(5) Suppose X and Y are independent random variables and that X is exponential with $\lambda=\frac{1}{4}$ and Y is uniform on $(2,5)$. Calculate the probability that $2X+Y<8$.

- **Solution:** We know that

$$f_X(x)=\begin{cases}\frac{1}{4}e^{-\frac{x}{4}} & \text{when } x\ge 0\\ 0 & \text{otherwise}\end{cases}$$

and

$$f_Y(y)=\begin{cases}\frac{1}{3} & \text{when } 2<y<5\\ 0 & \text{otherwise}\end{cases}$$

Since X,Y are independent then $f_{X,Y}=f_Xf_Y$, thus

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{12}e^{-\frac{x}{4}} & \text{when } x\ge 0, 2<y<5\\ 0 & \text{otherwise}\end{cases}$$

- Draw the region $(2X + Y < 8)$, which correspond to $0 \leq x$, $0 < y < 5$ and $y < 8-2x$. Drawing a picture of the region, we get the corresponding bounds of $2 < y < 5$ and $0 < x < 4 - \frac{y}{2}$, so that

$$\begin{aligned} \mathbb{P}(2X + Y < 8) &= \int_2^5 \int_0^{4-\frac{y}{2}} \frac{1}{12} e^{-\frac{x}{4}} dxdy \\ &= \int_2^5 \frac{1}{3}\left(1 - e^{y/8-1}\right) dx \\ &= 1 - \frac{8}{3}\left(e^{-\frac{3}{8}} - e^{-\frac{3}{4}}\right) \end{aligned}$$

(6) Consider X and Y given by the joint density

$$f(x,y) = \begin{cases} 10x^2 y & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal pdfs, $f_X(x)$ and $f_Y(x)$.
- **Solution:** We have

$$\begin{aligned} f_X(x) &= \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}, \\ f_Y(y) &= \begin{cases} \frac{10}{3} y(1-y^3) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

(b) Are X and Y independent random variables?
- **Solution:** No, since $f_{X,Y} \neq f_X f_Y$.

(c) Find $\mathbb{P}\left(Y \leq \frac{X}{2}\right)$.
- **Solution:** $\mathbb{P}\left(Y \leq \frac{X}{2}\right) = \frac{1}{4}$.

(d) Find $\mathbb{P}\left(Y \leq \frac{X}{4} \mid Y \leq \frac{X}{2}\right)$.
- **Solution:** Also $\frac{1}{4}$.

(e) Find $\mathbb{E}[X]$.
- **Solution:** Use f_X from part (a) the definition of expected value.

(7) Consider X and Y given by the joint density

$$f(x,y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the joint pdf's, f_X and f_Y.
- **Solution:** $f_X = 2x$ and $f_Y = 2y$.

(b) Are X and Y independent?
- **Solution:** Yes! Since $f(x,y) = f_X f_Y$.

(c) Find $\mathbb{E}Y$.
- **Solution:** We have $\mathbb{E}Y = \int_0^1 y \cdot 2y dy = \frac{2}{3}$.

(8) Consider X, Y given by the joint pdf

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent random variables?

- **Solution:** We get $f_X = \left(\frac{2}{3}x + \frac{2}{3}\right)$ while $f_Y = \frac{1}{3} + \frac{4}{3}y$ and $f \neq f_X f_Y$.

(9) Suppose that gross weekly ticket sales for UConn basketball games are normally distributed with mean \$2,200,000 and standard deviation \$230,000. What is the probability that the total gross ticket sales over the next two weeks exceeds \$4,600,000? What assumption did you use?

- **Solution:** If $W = X_1 + X_2$ is the sales over the next two weeks, then W is normal with mean $2,200,000 + 2,200,000 = 4,400,00$ and variance $230,000^2 + 230,000^2$. Thus the variance is $\sqrt{230,000^2 + 230,000^2} = 325,269.1193$. Here we used the assumption that X_1, X_2 are independent random variables, that is, basketball ticket sales are independent from week to week.
- Hence

$$\begin{aligned} \mathbb{P}(W > 5,000,000) &= \mathbb{P}\left(Z > \frac{4,600,000 - 4,400,000}{325,269.1193}\right) \\ &= \mathbb{P}(Z > .6149) \\ &\approx 1 - \Phi(.61) \\ &= .27 \end{aligned}$$

(10) Suppose the joint density function of the random variable X_1 and X_2 are

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_1 = 2X_1 + X_2$ and $Y_2 = X_1 - 3X_2$. What is the joint density function of Y_1 and Y_2?

- **Solution:**
  - Step1: Find the Jacobian: Note that

$$\begin{aligned} y_1 &= g_1(x_1, x_2) = 2x_1 + x_2, \\ y_2 &= g_2(x_1, x_2) = x_1 - 3x_2. \end{aligned}$$

So

$$J(x_1, x_2) = \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = -7.$$

  - Step2: Solve for x_1, x_2 and get

$$\begin{aligned} x_1 &= \frac{3}{7}y_1 + \frac{1}{y}y_2 \\ x_2 &= \frac{1}{7}y_1 - \frac{2}{7}y_2 \end{aligned}$$

  - Step3: The joint pdf of Y_1, Y_2 is given by the formula:

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(x_1, x_2)\,|J(x_1, x_2)|^{-1} \\ &= f_{X_1,X_2}\left(\frac{3}{7}y_1 + \frac{1}{y}y_2, \frac{1}{7}y_1 - \frac{2}{7}y_2\right)\frac{1}{7}. \end{aligned}$$

Since we are given the joint pdf of X_1 and X_2, then plugging it these into f_{X_1,X_2}, we have

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{4}{7^3}(3y_1 + y_2)(y_1 - 2y_2) & 0 < 3y_1 + y_2 < 7, 0 < y_1 - 2y_2 < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(11) Suppose the joint density function of the random variable X_1 and X_2 are

$$f(x_1, x_2) = \begin{cases} \frac{3}{2}\left(x_1^2 + x_2^2\right) & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_1 = X_1 - 2x_2$ and $Y_2 = 2X_1 + 3X_2$. What is the joint density function of Y_1 and Y_2?

- **Solution:**
  - Step1: Find the Jacobian: Note that

$$\begin{aligned} y_1 &= g_1(x_1, x_2) = x_1 - 2x_2, \\ y_2 &= g_2(x_1, x_2) = 2x_1 + 3x_2. \end{aligned}$$

    So

$$J(x_1, x_2) = \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = 7.$$

  - Step2: Solve for x_1, x_2 and get

$$\begin{aligned} x_1 &= \frac{1}{7}(3y_1 + 2y_2) \\ x_2 &= \frac{1}{7}(-2y_1 + y_2) \end{aligned}$$

  - Step3: The joint pdf of Y_1, Y_2 is given by the formula:

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(x_1, x_2)\left|J(x_1, x_2)\right|^{-1} \\ &= f_{X_1,X_2}\left(\frac{1}{7}(3y_1 + 2y_2), \frac{1}{7}(-2y_1 + y_2)\right)\frac{1}{7}. \end{aligned}$$

    Since we are given the joint pdf of X_1 and X_2, then plugging it these into f_{X_1,X_2}, we have

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{3}{2} \cdot \frac{1}{7^3}\left((3y_1 + 2y_2)^2 + (-2y_1 + y_2)^2\right) & 0 < 3y_1 + 2y_2 < 7, 0 < -2y_1 + y_2 < 7 \\ 0 & \text{otherwise.} \end{cases}$$

CHAPTER 12

Expectations

(1) Suppose the joint distribution for X and Y is given by the joint probability mass function shown below:

$Y \backslash X$	0	1
0	0	.3
1	.5	.2

. Calculate $\mathbb{E}[XY], \mathbb{E}[X], \mathbb{E}[Y]$.

- **Solution:** First let's calculate the marginal distributions:

$Y \backslash X$	0	1	
0	0	.3	.3
1	.5	.2	.7
	.5	.5	

Then

$$\begin{aligned} \mathbb{E}XY &= (0 \cdot 0)\, 0 + (0 \cdot 1)\, .5 + (1 \cdot 0)\, .3 + (1 \cdot 1)\, .2 = .2 \\ \mathbb{E}X &= 0 \cdot .5 + 1 \cdot .5 = .5 \\ \mathbb{E}Y &= 0 \cdot .3 + 1 \cdot .7 = .7. \end{aligned}$$

(2) Let X and Y be random variables whose joint probability density function is given by

$$f(x,y) = \begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\mathbb{E}[XY], \mathbb{E}X$, and $\mathbb{E}Y$.

- **Solution:** We'll need $\mathbb{E}[XY], \mathbb{E}X$, and $\mathbb{E}Y$:

$$\begin{aligned} \mathbb{E}[XY] &= \int_0^1 \int_0^1 xy\,(x+y)\, dydx = \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3} \right) dx = \frac{1}{3}, \\ \mathbb{E}X &= \int_0^1 \int_0^1 x(x+y)dydx = \int_0^1 (x^2 + \frac{x}{2})dx = \frac{7}{12} \\ \mathbb{E}Y &= \frac{7}{12}, \text{ by symmetry with the } \mathbb{E}X \text{ case.} \end{aligned}$$

(3) Let X be normally distributed with mean 1 and variance 9. Let Y be exponentially distributed with $\lambda = 2$. Suppose X and Y are independent. Find $\mathbb{E}\left[(X-1)^2 Y\right]$. (Hint: Use a properties about expectations)

- **Solution:** Since X, Y are independent then

$$\begin{aligned}\mathbb{E}\left[(X-1)^2 Y\right] &= \mathbb{E}\left[(X-1)^2\right]\mathbb{E}[Y] \\ &= \operatorname{Var}(X)\mu_Y \\ &= 9\cdot\frac{1}{2} \\ &= \frac{9}{2}.\end{aligned}$$

(4) Suppose $X \sim \text{Bern}\left(\frac{1}{2}\right)$ and $Y \sim \text{Exp}(2)$ are independent then calculate $\mathbb{E}\left[e^{XY}\right]$.
- **Solution:** Since X is discrete and Y is continuous then we compute using the mixed random variable formula, with $g(x,y) = e^{xy}$.
- Recall $X \in \{0,1\}$ with $p_X(0) = \frac{1}{2}$ and $p_X(1) = \frac{1}{2}$
- Recall $Y \in (0,\infty)$ with $f_Y(y) = 2e^{-2y}$, then
-

$$\begin{aligned}\mathbb{E}[g(X,Y)] &= \sum_{i=0}^{1}\int_0^\infty g(x_i,y)p_X(x)f_Y(y)dy \\ &= \int_0^\infty g(0,y)p_X(0)f_Y(y)dy + \int_0^\infty g(1,y)p_X(1)f_Y(y)dy \\ &= \int_0^\infty e^{0\cdot y}\left(\frac{1}{2}\right)\left(2e^{-2y}\right)dy + \int_0^\infty e^{1\cdot y}\left(\frac{1}{2}\right)\left(2e^{-2y}\right)dy \\ &= \int_0^\infty e^{-2y}dy + \int_0^\infty e^{y}e^{-2y}dy \\ &= \int_0^\infty e^{-2y}dy + \int_0^\infty e^{-y}dy \\ &= \frac{1}{2}+1 \\ &= \frac{3}{2}.\end{aligned}$$

(5) Suppose the joint distribution for X and Y is given by the joint probability mass function shown below:

$Y\backslash X$	0	1
0	0	.3
1	.5	.2

.

(a) Calculate the covariance of X and Y.

- **Solution:** First let's calculate the marginal distributions:

$Y\backslash X$	0	1	
0	0	.3	.3
1	.5	.2	.7
	.5	.5	

Then

$$\begin{aligned}\mathbb{E}XY &= (0\cdot 0)\,0 + (0\cdot 1)\,.5 + (1\cdot 0)\,.3 + (1\cdot 1)\,.2 = .2 \\ \mathbb{E}X &= 0\cdot .5 + 1\cdot .5 = .5 \\ \mathbb{E}Y &= 0\cdot .3 + 1\cdot .7 = .7.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}\left[XY\right]-\mathbb{E}X\mathbb{E}Y \\ &= .2-(.5)\,(.7)=-.15\end{aligned}$$

(b) Calculate $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.
- **Solution:** First need

$$\begin{aligned}\mathbb{E}X^2 &= 0^2.5+1^2.5=.5 \\ \mathbb{E}Y^2 &= 0^2.3+1^2.7=.7\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}X^2-(\mathbb{E}X)^2=.5-(.5)^2=.25, \\ \mathrm{Var}(Y) &= \mathbb{E}Y^2-(\mathbb{E}Y)^2=.7-(.7)^2=.21,\end{aligned}$$

(c) Calculate $\rho(X,Y)$.
- **Solution:**

$$\rho(X,Y)=\frac{\mathbf{Cov}(X,Y)}{\sqrt{\mathbf{Var}(X)\cdot\mathbf{Var}(Y)}}=\approx-.6547.$$

(6) Let X and Y be random variable whose joint probability density function is given by

$$f(x,y)=\begin{cases}x+y & 0<x<1,0<y<1\\ 0 & \text{otherwise.}\end{cases}$$

(a) Calculate the covariance of X and Y.
- **Solution:** We'll need $\mathbb{E}\left[XY\right],\mathbb{E}X$, and $\mathbb{E}Y$:

$$\begin{aligned}\mathbb{E}\left[XY\right] &= \int_0^1\int_0^1 xy\,(x+y)\,dydx=\int_0^1\left(\frac{x^2}{2}+\frac{x}{3}\right)dx=\frac{1}{3}, \\ \mathbb{E}X &= \int_0^1\int_0^1 x(x+y)dydx=\int_0^1(x^2+\frac{x}{2})dx=\frac{7}{12} \\ \mathbb{E}Y &= \frac{7}{12},\text{ by symmetry with the }\mathbb{E}X\text{ case.}\end{aligned}$$

Therefore,

$$\mathrm{Cov}(X,Y)=\mathbb{E}\left[XY\right]-\mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]=\frac{1}{3}-\left(\frac{7}{12}\right)^2=-\frac{1}{144}.$$

(b) Calculate $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.
- **Solution:** We need $\mathbb{E}X^2$ and $\mathbb{E}Y^2$,

$$\mathbb{E}X^2=\int_0^1\int_0^1 x^2(x+y)dydx=\int_0^1(x^3+\frac{x^2}{2})dx=\frac{5}{12}$$

so we know that $\mathbb{E}Y^2=\frac{5}{12}$ by symmetry. Therefore

$$\mathrm{Var}(X)=\mathrm{Var}(Y)=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}.$$

(c) Calculate $\rho(X,Y)$.

- **Solution:**

$$\rho(X,Y) = \frac{\mathbf{Cov}(X,Y)}{\sqrt{\mathbf{Var}(X) \cdot \mathbf{Var}(Y)}} = -\frac{1}{11}.$$

CHAPTER 13

Moment generating functions

(1) Suppose that you have a fair 4-sided die, and let X be the random variable representing the value of the number rolled.

(a) Write down the moment generating function for X.

- **Solution:**

$$\begin{aligned} m_X(t) &= \mathbb{E}\left[e^{tX}\right] = e^{1\cdot t}\frac{1}{4} + e^{2\cdot t}\frac{1}{4} + e^{3\cdot t}\frac{1}{4} + e^{4\cdot t}\frac{1}{4} \\ &= \frac{1}{4}\left(e^{1\cdot t} + e^{2\cdot t} + e^{3\cdot t} + e^{4\cdot t}\right) \end{aligned}$$

(b) Use this moment generating function to compute the first and second moments of X.

- **Solution:** We have

$$\begin{aligned} m'_X(t) &= \frac{1}{4}\left(e^{1\cdot t} + 2e^{2\cdot t} + 3e^{3\cdot t} + 4e^{4\cdot t}\right), \\ m'_X(t) &= \frac{1}{4}\left(e^{1\cdot t} + 4e^{2\cdot t} + 9e^{3\cdot t} + 16e^{4\cdot t}\right), \end{aligned}$$

so

$$\mathbb{E}X = m'_X(0) = \frac{1}{4}(1+2+3+4) = \frac{5}{2}$$

and

$$\mathbb{E}X^2 = m''_X(0) = \frac{1}{4}(1+4+9+16) = \frac{15}{2}.$$

(2) Let X be a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} e^{-2x} + \frac{1}{2}e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Write down the moment generating function for X.

- **Solution:**

$$\begin{aligned} m_X(t) &= \mathbb{E}\left[e^{tX}\right] = \int_0^\infty e^{tx}\left(e^{-2x} + \frac{1}{2}e^{-x}\right) dx \\ &= \frac{1}{2-t} + \frac{1}{2(1-t)}, \quad \text{for } t < 1. \end{aligned}$$

(b) Use this moment generating function to compute the first and second moments of X.

- **Solution:** We have

$$\begin{aligned} m'_X(t) &= \frac{1}{(2-t)^2} + \frac{1}{2(1-t)^2} \\ m''_X(t) &= \frac{2}{(2-t)^3} + \frac{1}{(1-t)^3} \end{aligned}$$

so $\mathbb{E}X = m_X'(0) = \frac{3}{4}$ and $\mathbb{E}X^2 = m_X'' = \frac{5}{4}$.

(3) Suppose X and Y are Poisson independent random variables with parameters λ_x, λ_y, respectively. Find the distribution of $X+Y$.

- **Solution:** Since $X \sim \text{Pois}(\lambda_x)$ and $Y \sim \text{Pois}(\lambda_y)$ then

$$m_X(t) = e^{\lambda_x\left(e^t-1\right)} \text{ and } m_Y(t) = e^{\lambda_y\left(e^t-1\right)}.$$

Then

$$\begin{aligned} m_{X+Y}(t) &= m_X(t)m_Y(t), \quad \text{by independence} \\ &= e^{\lambda_x\left(e^t-1\right)}e^{\lambda_y\left(e^t-1\right)} \\ &= e^{(\lambda_x+\lambda_y)\left(e^t-1\right)}. \end{aligned}$$

Thus by the Theorem in Section 13.1, $X+Y \sim \text{Pois}\left(\lambda_x+\lambda_y\right)$.

(4) True or False? Suppose $X \sim \text{Exp}\left(\lambda_x\right)$ and $Y \sim \text{Exp}\left(\lambda_y\right)$ with parameters $\lambda_x, \lambda_y > 0$ and suppose X, Y are independent. Is $X+Y \sim \text{Exp}\left(\lambda_x+\lambda_y\right)$?

- **Solution:** Since $X \sim \text{Exp}\left(\lambda_x\right)$ and $Y \sim \text{Exp}\left(\lambda_y\right)$ then

$$m_X(t) = \frac{\lambda_x}{\lambda_x - t}, \text{for } t < \lambda_x \text{ and } m_X(t) = \frac{\lambda_y}{\lambda_y - t}, \text{for } t < \lambda_x.$$

Then

$$\begin{aligned} m_{X+Y}(t) &= m_X(t)m_Y(t), \quad \text{by independence} \\ &= \frac{\lambda_x}{\lambda_x - t}\frac{\lambda_y}{\lambda_y - t} \\ &= \frac{\lambda_x\lambda_y}{\left(\lambda_x - t\right)\left(\lambda_y - t\right)}. \end{aligned}$$

While if $Z \sim \text{Exp}\left(\lambda_x+\lambda_y\right)$ then

$$m_Z(t) = \frac{\lambda_x+\lambda_y}{\lambda_x+\lambda_y-t}, \text{for } t < \lambda_x+\lambda_y,$$

but since

$$m_{X+Y}(t) = \frac{\lambda_x\lambda_y}{\left(\lambda_x - t\right)\left(\lambda_y - t\right)} \neq \frac{\lambda_x+\lambda_y}{\lambda_x+\lambda_y-t} = m_Z(t)$$

for general λ_x, λ_y then the answer is false!.

(5) Suppose that a mathematician determines that the revenue the UConn Dairy Bar makes in a week is a random variable, X, with moment generating function

$$M_X(t) = \frac{1}{(1-2500t)^4}.$$

Calculate the standard deviation of the revenue the UConn Dairy bar makes in a week.

- **Solution:** We want $SD(X) = \sqrt{\text{Var}(X)}$. But $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$. We compute

$$\begin{aligned} m'(t) &= 4\,(2500)\,(1-2500t)^{-5}, \\ m''(t) &= 20\,(2500)^2\,(1-2500t)^{-6}, \\ \mathbb{E}X &= m'(0) = 10,000 \\ \mathbb{E}X^2 &= m''(0) = 125,000,000 \\ \text{Var}(X) &= 125,000,000 - 10,000^2 = 25,000,000 \\ SD(X) &= \sqrt{25,000,000} = \mathbf{5,000}. \end{aligned}$$

(6) Let X and Y be two independent random variables with respective moment generating functions

$$m_X(t) = \frac{1}{1-5t}, \quad \text{if } t < \frac{1}{5}, \quad m_Y(t) = \frac{1}{(1-5t)^2}, \quad \text{if } t < \frac{1}{5}.$$

Find $\mathbb{E}\,(X+Y)^2$.

- **Solution:** First recall that if we let $W = X + Y$ t, and X, Y indepedent then

$$m_W(t) = m_{X+Y}(t) = m_X(t) m_Y(t) = \frac{1}{(1-5t)^3},$$

recall that $\mathbb{E}\left[W^2\right] = m_W''(0)$. Thus we need to compute some derivatives

$$\begin{aligned} m_W'(t) &= \frac{15}{(1-5x)^4}, \\ m_W''(t) &= \frac{300}{(1-5x)^5}, \end{aligned}$$

thus

$$\mathbb{E}\left[W^2\right] = m_W''(0) = \frac{300}{(1-0)^5} = 300.$$

(7) Suppose $X \sim \text{Exp}\,(2)\,, Y \sim \text{Bern}\left(\frac{1}{2}\right), Z \sim \text{Exp}\,(1)$. Suppose X, Y, Z are independent and define the random variable

$$W = X + YZ.$$

Compute the moment generating function of W and find the distribution of W exactly. (Hint: Simplify as much as possible and your answer will be one of the known distributions in the distribution table.)

- **Solution:**
- We first compute

$$\begin{aligned} m_W(t) &= m_{X+YZ}(t) \\ &= m_X(t)\, m_{YZ}(t)\,, \text{ by independence. } (\star) \end{aligned}$$

- Now since $X \sim \text{Exp}\,(2)$ we already know that

$$m_X(t) = \frac{2}{2-t}.$$

- So we need to compute $m_{YZ}(t)$ directly, since the MGF of YZ is not known:

$$m_{YZ}(t) = \mathbb{E}\left[e^{tYZ}\right]$$

- Since Y is discrete and Z is continuous then we compute using the mixed random variable formula from Section 12.1, with $g(y,z) = e^{t\cdot yz}$.
- Recall $Y \in \{0,1\}$ with $p_Y(0) = \frac{1}{2}$ and $p_Y(1) = \frac{1}{2}$
- Recall $Z \in (0,\infty)$ with $f_Z(z) = e^{-z}$, then
- Now

$$\begin{aligned}
\mathbb{E}\left[g(Y,Z)\right] &= \sum_{i=0}^{1}\int_0^\infty g(y_i,z)p_Y(y)f_Z(z)dz \\
&= \int_0^\infty g(0,z)p_Y(0)f_Z(z)dz + \int_0^\infty g(1,z)p_Y(1)f_Z(z)dz \\
&= \int_0^\infty e^{t\cdot 0\cdot z}\left(\frac{1}{2}\right)\left(e^{-z}\right)dz + \int_0^\infty e^{tz}\left(\frac{1}{2}\right)\left(e^{-z}\right)dz \\
&= \frac{1}{2}\int_0^\infty e^{-z}dz + \frac{1}{2}\int_0^\infty e^{(t-1)z}dz \\
&= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{1-t} \\
&= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{1-t}
\end{aligned}$$

- Finally, putting this back into $(\star)$we have

$$\begin{aligned}
m_W(t) &= m_X(t)\, m_{YZ}(t) \\
&= \frac{2}{2-t}\left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{1-t}\right) \\
&= \frac{2}{2-t}\left(\frac{1-t}{2(1-t)} + \frac{1}{2}\cdot\frac{1}{1-t}\right) \\
&= \frac{2}{2-t}\left(\frac{1-t}{2(1-t)} + \frac{1}{2(1-t)}\right) \\
&= \frac{2}{2-t}\left(\frac{2-t}{2(1-t)}\right) \\
&= 2\left(\frac{1}{2(1-t)}\right) \\
&= \frac{1}{1-t}.
\end{aligned}$$

- Thus after simplifying,

$$m_W(t) = \frac{1}{1-t},$$

which is the MGF of an $\mathrm{Exp}(1)$, thus

$$W \sim \mathrm{Exp}(1).$$

CHAPTER 14

Limit Laws

(1) In a 162-game season, find the approximate probability that a team with a 0.5 chance of winning will win at least 87 games.

- **Solution:** Let X_i be 1 if the team win's the ith game and 0 if the team loses. This is a Bernoulli R.V. with $p = .5$. Thus $\mu = p = .5$ and $\sigma^2 = p(1-p) = (.5)^2$. Then

$$X = \sum_{i=1}^{162} X_i$$

is the number of games won in the season. Using CLT

$$\begin{aligned} n\mu &= 162 \cdot .5 = 81 \\ \sigma\sqrt{n} &= .5\sqrt{162} = 6.36 \end{aligned}$$

then

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{162} X_i \geq 87\right) &= \mathbb{P}(X \geq 86.5) \\ &= \mathbb{P}\left(\frac{X-81}{6.36} > \frac{86.5-81}{6.36}\right) \\ &\approx \mathbb{P}(Z > .86) = .1949 \end{aligned}$$

where we used a correction since X is a discrete r.v.

(2) An individual students MATH 3160 Final exam score at UConn is a random variable with mean 75 and variance 25, How many students would have to take the examination to ensure with probability at least .9 that the class average would be within 5 of 75?

- **Solution:** Now $\mu = 75$, $\sigma^2 = 25, \sigma = 5$.

$$
\begin{aligned}
\mathbb{P}\left(70 < \frac{\sum_{i=1}^{n} X_i}{n} < 80\right) \geq .9 \quad & \Longleftrightarrow \quad \mathbb{P}\left(70 \cdot n < \sum_{i=1}^{n} X_i < 80 \cdot n\right) \geq .9 \\
& \Longleftrightarrow \quad \mathbb{P}\left(\frac{70 \cdot n - 75 \cdot n}{5\sqrt{n}} < Z < \frac{80 \cdot n - 75 \cdot n}{5\sqrt{n}}\right) \geq .9 \\
& \Longleftrightarrow \quad \mathbb{P}\left(-5\frac{\sqrt{n}}{5} < Z < 5\frac{\sqrt{n}}{5}\right) \geq .9 \\
& \Longleftrightarrow \quad \mathbb{P}\left(-\sqrt{n} < Z < \sqrt{n}\right) \geq .9 \\
& \Longleftrightarrow \quad \Phi\left(\sqrt{n}\right) - \Phi\left(-\sqrt{n}\right) \geq .9 \\
& \Longleftrightarrow \quad \Phi\left(\sqrt{n}\right) - \left(1 - \Phi\left(\sqrt{n}\right)\right) \geq .9 \\
& \Longleftrightarrow \quad 2\Phi\left(\sqrt{n}\right) - 1 \geq .9 \\
& \Longleftrightarrow \quad \Phi\left(\sqrt{n}\right) \geq .95
\end{aligned}
$$

Using the table inversely we have that

$$\sqrt{n} \geq 1.65 \implies n \geq 2.722$$

hence the first integer that insurers that $n \geq 2.722$ is

$$n = 3.$$

(3) Let $X_1, X_2, \ldots, X_{100}$ be independent exponential random variables with parameter $\lambda = 1$. Use the central limit theorem to approximate

$$\mathbb{P}\left(\sum_{i=1}^{100} X_i > 90\right).$$

- **Solution:** Since $\lambda = 1$ then $\mathbb{E}X_i = 1$ and $\mathrm{Var}(X_i) = 1$. Use CLT

$$
\begin{aligned}
n\mu &= 100 \cdot 1 = 100 \\
\sigma\sqrt{n} &= 1 \cdot \sqrt{100} = 10.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{100} X_i > 90\right) &= \mathbb{P}\left(\frac{\sum_{i=1}^{100} X_i - 100 \cdot 1}{1 \cdot \sqrt{100}} > \frac{90 - 100 \cdot 1}{1 \cdot \sqrt{100}}\right) \\
&\approx \mathbb{P}(Z > -1) = .8413.
\end{aligned}
$$

(4) Suppose an insurance company has 10,000 automobile policy holders. The expected yearly claim per policy holder is \$240, with a standard deviation of \$800. Approximate the probability that the total yearly claim is greater than \$2,500,000.

- **Solution:**

$$
\begin{aligned}
\mathbb{P}(X \geq 1300) &= \mathbb{P}\left(\frac{X - 2400000}{80000} \geq \frac{2500000 - 2400000}{80000}\right) \\
&\approx \mathbb{P}(Z \geq 1.25) \\
&= 1 - \Phi(1.25) = 1 - .8944 \\
&= .1056.
\end{aligned}
$$

(5) Suppose you are the only clerk at the UConn dairy bar. Suppose that the checkout time at the dairy bar has a mean of 5 minutes and a standard deviation of 2 minutes. Estimate the probability that a clerk will serve at least 36 customers during her 3-hour and a half shift.

- **Solution:** Let X_i be the time it takes to check out customer i. Then
$$X = X_1 + \cdots + X_{36}$$
is the time it takes to check out 36 customer. We want $\mathbb{P}(X \leq 210)$.
- Use CLT,
$$\begin{aligned} n\mu &= 36 \cdot 5 = 180 \\ \sigma\sqrt{n} &= 2\sqrt{36} = 12. \end{aligned}$$
- Thus
$$\begin{aligned} \mathbb{P}(X \leq 210) &= \mathbb{P}\left(\frac{X-180}{12} \leq \frac{210-180}{12}\right) \\ &\approx \mathbb{P}(Z \leq 2.5) \\ &= \Phi(2.5) \\ &= .9938. \end{aligned}$$

(6) Shabazz Napier is a basketball player in the NBA. His expected number of points per game is 15 with a standard deviation of 5 points per game. The NBA season is 82 games long. Shabazz is guaranteed a ten million dollar raise next year if he can score a total of 1300 points this season. Approximate the probability that Shabazz will get a raise next season.

- **Solution:**Let X_i be the number of points scored by Shabazz in game i . Then
$$X = X_1 + \cdots + X_{82}$$
is the total number of points in a whole season. We want $\mathbb{P}(X \geq 1800)$.
- Use CLT,
$$\begin{aligned} n\mu &= 82 \cdot 15 = 1230 \\ \sigma\sqrt{n} &= 5\sqrt{82} = 45.28. \end{aligned}$$
- Thus
$$\begin{aligned} \mathbb{P}(X \geq 1300) &= \mathbb{P}\left(\frac{X-1230}{45.28} \geq \frac{1300-1230}{45.28}\right) \\ &\approx \mathbb{P}(Z \geq 1.55) \\ &= 1-\Phi(1.55) = 1-.9394 \\ &= .0606. \end{aligned}$$